THE THIRD REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE third regular meeting of the Southwestern Section of the Society was held at the University of Missouri, Columbia, Missouri, on Saturday, November 27, 1909. The following members of the Society were present:

Professor L. D. Ames, Professor W. C. Brenke, Professor E. W. Davis, Professor G. R. Dean, Professor L. M. Defoe, Mr. A. B. Frizell, Mr. E. S. Haynes, Professor E. R. Hedrick, Dr. Louis Ingold, Professor O. D. Kellogg, Professor W. H. Roever, Dr. Mary S. Walker, Dr. Paul Wernicke, and Professor W. D. A. Westfall.

The morning session was opened at 10 A.M. and the afternoon session at 2 P.M., Professor Davis presiding. Lincoln, Nebraska, was decided upon for the next meeting, and the following program committee was elected: Professor Davis (chairman), Dr. Wernicke, Professor Kellogg (secretary).

The following papers were presented:

(1) Professor G. O. JAMES: "On the reduction of time observations in vertical circle through Polaris."

(2) Professor L. D. AMES: "Some theorems of Lie on ordinary differential equations."

(3) Professor J. B. SHAW: "Scalars of lineal products."

(4) Professor M. B. PORTER: "On Fourier sequences."

(5) Dr. PAUL WERNICKE: "Note on sine theorems in hyperspace."

(6) Professor W. H. ROEVER: "The southerly deviation of falling bodies."

(7) Professor G. R. DEAN: "Stresses in an isotropic plate."

(8) Professor E. W. DAVIS: "A paradox relating to the imaginary line."

(9) Professor W. D. A. WESTFALL: "The continuity of integral rational functions of infinitely many variables."

(10) Professor H. B. NEWSON: "On linear groups in two variables."

(11) Professor W. C. BRENKE: "Summation of a series of Bessel's functions by means of an integral."

(12) Professor E. R. HEDRICK: "On a property of assemblages whose derivatives are closed."

(13) Mr. A. B. FRIZELL: "Natural numbers defined by the principles of abstract groups."

(14) Dr. LOUIS INGOLD: "Outline of a vector theory of curves."

In the absence of the authors, the papers of Professors James, Shaw, and Newson were read by title. Abstracts of the papers follow below.

1. From the rigorous formulas for the reduction of time observations in the vertical circle through Polaris Professor James develops the chronometer correction in a power series in p_0, the polar distance of Polaris, neglecting terms of the third and higher powers in p_0. By the introduction of the reduction to the pole, published each year in the *American Ephemeris and Nautical Almanac*, this series is transformed into one in which the term in p_0^2 is missing, thus reducing the approximate formula to a single term. The error of this simple formula is investigated and latitudes for which it is admissible determined. An analogous development and transformation is made for azimuth, and the resulting reduction formulas are in both cases adapted to the engineer's transit.

2. Consider an ordinary differential equation $Xy' - Y = 0$ from the standpoint of the Lie theory. Consider the three following statements: (A) Its integral curves admit the group

$$Uf = \xi \frac{\partial t}{\partial x} + \eta \frac{\partial t}{\partial y}.$$

(B) The equation itself admits the extended group. (C) An integrating factor is $1/(X\eta - Y\xi)$. Professor Ames's paper states three well-known theorems: (1) that (B) and (C) are equivalent, (2) that (A) and (B) are equivalent, (3) that (A) and (C) are equivalent. Theorem (1) was proved in a previous paper. The other two are proved in the present paper. The statements and proofs are given in analytic terms without making any use of the group theory, and the three theorems are proved independently of one another. Of course any one follows from the other two.

3. A lineal product as defined by Grassmann is one whose properties are independent of the ground or system of units in

terms of which the numbers are expressed. Scalars of such products would have the same property. Thus for quaternions $S\cdot\alpha\beta\gamma = -S\cdot\beta\alpha\gamma$, whatever numbers α, β, γ may be. Scalars of lineal products must therefore be expressible in terms of more elementary scalars of lineal products, or else be themselves elementary scalars.

Professor Shaw's paper discusses the scalar defined by the recurrence formula $I\cdot\alpha_1\alpha_2\cdots\alpha_{2m} = I\cdot\alpha_1\alpha_2 I\cdot\alpha_3\cdots\alpha_{2m} + I\cdot\alpha_1\alpha_3 I\cdot\alpha_4\cdots\alpha_{2m}\alpha_2 + I\cdot\alpha_1\alpha_4 I\cdot\alpha_5\cdots\alpha_{2m}\alpha_2\alpha_3 + \cdots + I\cdot\alpha_1\alpha_{2m} I\cdot\alpha_2\cdots\alpha_{2m-1}$. This scalar is a Pfaffian and is of much use. Scalars of corresponding types of the forms $I\cdot\alpha_1\cdots\alpha_{3m} = \Sigma\cdot I\cdot\alpha_1\alpha_2\alpha_3\cdot I\cdot\alpha_4\cdots\alpha_{3m}$, etc., are slightly examined.

4. If

$$\sum_1^n (a_n \sin nx + b_n \cos nx) = S_n$$

be the Fourier sequence of order n of the function $f(x)$ when it is supposed that

$$\int_{-\pi}^{\pi} [f(x)]^2 dx$$

exists, Professor Porter shows that from any sequence $S_{n_i}\,(i = 1, 2, \cdots)$ a sequence can be picked out which converges over a point set of measure 2π to the value $f(x)$.

5. The formula used for the volume belonging to the n-dimensional "simplex" on page 212 of the February, 1909, BULLETIN $V = p_{01}p_{12}\sin(P_{01}, P_{12})\cdot p_{23}\sin(P_{12}, P_{23})\cdot\sin(P_{012}, P_{123})\cdots$ is capable of extension. It remains correct

1) if we replace the p_{ij} by the n joins p_{0j} of one point P_0 to the remaining n; and simultaneously $P_{ijk}, \cdots$ by P_{0jk};

2) if, after permuting the p_{ij} (and P_{ij}), we replace $P_{0123\cdots\nu}$ by any ν-plane parallel to the first, second, $\cdots$, νth P_{ij} (in the new order);

3) if we permute the vertices, hence the indices $0, 1, \cdots, n$, throughout.

Equating expressions thus recognized to be equal, Dr. Wernicke obtains theorems connecting the sines. For example, in 3-space,

$$p_{12}\sin(P_{01}, P_{12})\sin(P_{012}, P_{123}) = p_{13}\sin(P_{01}, P_{13})\sin(P_{013}, P_{132})$$

which, when $p_{12} = p_{13}$, becomes the sine theorem of spherical trigonometry.

As a further example, in 4-space, the angles made by the 2-planes P_{012} and P_{034} are connected with those made by P_{014} and P_{023} by the sine theorem

$$\sin(P_{01}, P_{02}) \sin(P_{03}, P_{04}) \sin_1(P_{012}, P_{034}) \sin_2(P_{012}, P_{034}) = \sin(P_{01}, P_{04}) \sin(P_{02}, P_{03}) \sin_1(P_{013}, P_{024}) \sin_2(P_{013}, P_{024}),$$

$\sin_1$, $\sin_2$ denoting the sines of their "first" and "second" angles, respectively. The paper will be offered to the *Annals of Mathematics* for publication.

6. A body dropped from a point P_0, attached to the rotating earth, falls in a certain curve c, in a field of force F_1, which is fixed in space. It receives an initial velocity which is the same as the velocity of P_0 at the instant it is dropped. A plumb-line supported at P_0 hangs in a field of force F_2 which rotates with the earth. If $U_1 = f_1(r, z)$ and $U_2 = f_2(r, z)$ (where r and z denote the distances of a general point from the axis of rotation OZ and the plane of the equator of the earth) are force functions of the fields F_1 and F_2 respectively,

$$U_1 + \tfrac{1}{2}\omega^2 r^2 = U_2,$$

in which ω denotes the angular velocity of the earth's rotation.

Professor Roever reduces the problem to one in two dimensions by the following device: The curve c, when rotated about the earth's axis, generates a surface of revolution of axis OZ. The meridian curve of this surface which lies in the plane of OZ and P_0 we call curve (1). Plumb-lines of different lengths supported at P_0 do not coincide. The locus of the plumb-bobs of all plumb-lines supported at P_0, which lies in the plane of OZ and P_0, we call curve (2). Curves (1) and (2) are tangent at P_0 to the line of force of the field F_2 which passes through P_0. In order to establish the existence of a southerly deviation (in the northern hemisphere) it is enough to show that curve (1) lies south of curve (2). The magnitude of the deviation is the distance between the points in which these curves pierce the equipotential surface $U_2 = K$ which represents the earth's surface.

In particular, the form of the function U_1 was assumed to be M/ρ, where M is the mass of the earth and ρ is the distance of a general point from the earth's center O. For this function the equipotential surface $K = U_2 = M/\rho + \frac{1}{2}\omega^2 r^2$, which is taken

to represent the earth's surface, is approximately a spheroid of ellipticity .0017. (That of the earth is about .0034). The common tangent and the common normal at P_0 of the curves (1) and (2) being chosen as the axes of ξ and y, the difference between the ordinates of the curves (1) and (2) is $A\xi^2 + B\xi^3 + \cdots$, where

$$A = \tfrac{1}{2}\frac{\sin 2\phi_0}{\rho_0}\left\{ 8\frac{g_1^2}{g_2^3}\rho_0\omega^2 - \frac{g_1}{g_2^2}\rho_0^2\omega^4(8\cos^2\phi_0 + 3\sin^2\phi_0)\right\},$$

in which ϕ_0 is geocentric latitude of P_0, ρ_0 is ρ for P_0, g_1 and g_2 are accelerations at P_0 due to the fields F_1 and F_2 respectively. Approximately $g_1^2/g_2^3\rho_0\omega^2 = 1/(17)^2$ and $g_1/g_2^3\rho_0^2\omega^4 = 1/(17)^4$. The expression gives the southerly deviation when ξ is the height of P_0.

7. Professor Dean considers a special case of the problem in elasticity known as the problem of Clebsch, the forces being parallel to the faces of the plate and distributed continuously across the edges. The assumption of Clebsch, that certain shearing stresses are zero, restricts the solution to the case of a plate of infinitesimal thickness and is not made here. It is shown in the standard treatises that the strain perpendicular to the planes of the faces is a linear function of the coordinates parallel to the faces. From this fact and the generalized form of Hooke's law it is shown that the cubical expansion is a linear function of two variables, and that the shears are proportional to the distance from the mid-section of the plate parallel to the faces. This function substituted in the displacement equations gives two Poisson equations by means of which the displacements are expressed as logarithmic potential functions. The stresses being functions of the derivatives of the displacements are easily determined in terms of the coordinates and three arbitrary constants, which may be determined from the boundary conditions of stress. The paper will be offered to the *Philosophical Magazine*.

8. Professor Davis shows that, taking two vectors as representative of two imaginary points, it is possible to pass through them an infinite number of imaginary lines. This is effected by proper projective changes in the system of coordinates. It is shown also that lines in different systems may have an infinite set of elements in common. The paper will be embodied in one to appear in the *Nebraska University Studies*.

9. In Professor Westfall's paper the continuity and boundedness of integral rational functions of infinitely many variables in a domain D_m: $\sum_{i=1}^{i=\infty} x_i^2 \leqq m$, is discussed. If $\lim f(x) = f(a)$ whenever $\lim \sum (x_i - a_i)^2 = 0$, for one point a of D_m, this holds for every point a, and the integral rational function is bounded in any given domain D_m. If $\lim f(x) = f(a)$ when $\lim x_i = a_i (i = 1, 2, 3, \cdots, +\infty)$ when $\sum_{i=1}^{i=\infty} x_i^2 \leqq m$ and $\sum_{i=1}^{i=\infty} a_i^2 < m$, this again holds true for any point x in any domain D_m. That is, semicontinuity or continuity at a single interior point is sufficient to ensure the same property at every point of any domain D_m in the case of integral rational functions.

10. In a paper presented at the summer meeting of the Society at Princeton, September 14, 1909, Professor Newson announced a fundamental theorem in the theory of linear groups. His present paper is devoted to the application of this theorem to the theory of linear groups in two variables. He is able to find all the continuous groups as previously determined by Lie, and all of the discontinuous groups as determined by Klein and also by Gordan.

11. By means of the known relation $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$ Professor Brenke obtains a linear differential equation in S_k, where S_k denotes the sum of the first k terms of the series. On solving for S_k and passing to the limit there results the equation

$$\sum_1^\infty c^n J_n(x) = \tfrac{1}{2} e^{ax} \int_0^x [cJ_0(x) + J_1(x)] e^{-ax} dx\,; \quad \left[a = \frac{c^2 - 1}{c}\right].$$

The result holds for all values of x and c.

12. Fréchet proved, in his thesis, that the Heine-Borel theorem holds for countable families in any compact assemblage in which "limit" is definable by means of "distance" in the sense in which those terms are there used. Professor Hedrick points out in the present paper that the same theorem is true for any compact, closed assemblage whose derivative is closed.

13. Mr. Frizell defines two rules of combination of symbols,

— a higher rule distributive according to the inductive formulas

$$(1)\quad (a \circ \mu)b = ab \circ \mu b, \cdots, \qquad (2)\quad a(b \circ \mu) = ab \circ a\mu, \cdots,$$

over a lower rule associative according to the inductive formula

$$(3)\qquad a \circ (b \circ \mu) = a \circ b \circ \mu, \cdots;$$

postulates two classes of symbols both possessing the fundamental group property as regards the rule denoted by the sign $\circ$, of which (4) one shall contain the symbol μ, and (5) the other shall contain the symbol $\mu\mu$, where (6) $\mu_i \circ \mu_j \neq \mu_i \cdots$; and demands (7) that the two classes shall be identical.

It follows that the well ordered infinite class defined by the above seven postulates constitutes an abelian semigroup with regard to each rule of combination and has a modulus for the higher rule, which is distributive over the lower.

14. In Dr. Ingold's paper an effort is made to determine how far arbitrary relations may be assigned between a tangent vector T (supposed to be a function of a parameter s) and its derivatives T', T'', T''', $\cdots$. A set of normal vectors N_1, N_2, N_3, $\cdots$ are defined by the relations

$$N_i = A_{i0}N_0 + A_{i1}N_1 + \cdots + A_{ii-1}N_{i-1} + A_{ii}T^{(i)},$$

where the scalar coefficients A_{ij} are so defined that the N_i form an orthogonal set of unit vectors.

It is shown first that the A_{ij} are expressible in terms of i quantities $1/r_i$ and their derivatives; that the $1/r_i$ are the curvatures of the curve thus defined; finally that the A_{ij} certainly exist if T is explicitly given, since the r_i, and hence also the A_{ij}, can be expressed in terms of T and its derivatives.

O. D. KELLOGG,
Secretary of the Section.

NOTE ON A NEW NUMBER THEORY FUNCTION.

BY MR. R. D. CARMICHAEL.

(Read before the American Mathematical Society, September 13, 1909.)

THE present note deals with the properties of a number theory function defined by means of Euler's ϕ-function in the following way:

$\lambda(p^a) = \phi(p^a)$ when p is an odd prime;

$\lambda(2^a) = \phi(2^a)$ if $a = 0$, 1, or 2; $\lambda(2^a) = \frac{1}{2}\phi(2^a)$ if $a > 2$;* $\lambda(2^a p_1^{a_1} \cdots p_i^{a_i}) =$ the lowest common multiple of $\lambda(2^a)$, $\lambda(p_1^{a_1})$, $\cdots$, $\lambda(p_i^{a_i})$, p_1, $\cdots$, p_i being different odd primes.

Throughout, in a congruence such as

$$x^a \equiv 1 \pmod{n}$$

it will be assumed that x is prime to n. Then we have the theorem

(1) $$x^{\lambda(p^a)} \equiv 1 \pmod{p^a}$$

for every prime p and integer a. For, by Fermat's theorem, (1) is true when p is an odd prime and also when $p = 2$ and $a = 1$ or 2, in view of the definition of λ. Then we have to examine only the case where $p = 2$ and $a > 2$.

Now by Fermat's theorem we have

$$x^{\phi(2^a)} \equiv 1 \pmod{2^a}, \quad (a > 2).$$

But it is known that the foregoing congruence has no primitive root; that is, for any odd x the congruence is true when $\phi(2^a)$ is replaced by some factor of $\phi(2^a)$ less than the number itself. But $\frac{1}{2}\phi(2^a) = \lambda(2^a)$ is the largest factor of $\phi(2^a)$ less than itself and contains all other such factors. Then

$$x^{\frac{1}{2}\phi(2^a)} \equiv 1 \pmod{2^a}, \quad (a > 2).$$

Hence the theorem of congruence (1) is proved.

This result may be employed to obtain a simple demonstration of the following analog of Fermat's general theorem:

* It is in respect to this part of the definition alone that $\lambda(n)$ differs from $\psi(n)$ defined by Bachmann, Niedere Zahlentheorie, I, p. 157.

I. *For any given n the congruence*

$$x^{\lambda(n)} \equiv 1 \pmod{n}$$

is satisfied by every x prime to n.

For suppose

$$n = 2^{\alpha} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_i^{\alpha_i}$$

and let β be any number prime to n. Then since $\lambda(n)$ is a multiple of every $\lambda(p^{\alpha})$, $p = 2, p_1, \cdots, p_i$, we have

$$\beta^{\lambda(n)} \equiv 1 \pmod{2^{\alpha}}, \qquad \beta^{\lambda(n)} \equiv 1 \pmod{p_1^{\alpha_1}},$$

$$\cdots, \beta^{\lambda(n)} \equiv 1 \pmod{p_i^{\alpha_i}}.$$

From these congruences the theorem follows.

If $a^{\lambda(n)}$ is the first power of a congruent to 1 modulo n, we may say that a is a primitive λ-root (mod n). To distinguish, we may speak of the usual primitive root as a primitive ϕ-root (mod n). It follows immediately from the theory of primitive ϕ-roots that primitive λ-roots always exist when n is the power of any prime; for this is but another statement of well-known results for the modulus p^{α}. The λ-function introduces a simplification and allows the principal theory of the existence of primitive roots to be summarized into the following theorem:

II. *In every congruence*

$$x^{\lambda(n)} \equiv 1 \pmod{n} \tag{2}$$

a solution g exists which is a primitive λ-root, and for any such solution g there are $\phi\{\lambda(n)\}$ primitive roots congruent to powers of g.

If any primitive root g exists, g^{α} is or is not a primitive root according as α is or is not prime to $\lambda(n)$; and therefore the number of primitive λ-roots which are congruent to powers of any such root g is $\phi\{\lambda(n)\}$.

The existence of a primitive λ-root in every case is easily shown by induction. If n is a power of a prime the theorem has already been established. We will suppose that it has been established when n is the product of powers of r different primes and show that the theorem still remains true when n is the product of powers of $r + 1$ different primes; and from this follows the theorem in general.

Put

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r} p_{r+1}^{a_{r+1}},$$

and let h be a primitive λ-root of

$$x^{\lambda(p_1^{a_1} \cdots p_r^{a_r})} \equiv 1 \pmod{p_1^{a_1} \cdots p_r^{a_r}}; \tag{3}$$

whence $h + p_1^{a_1} \cdots p_r^{a_r} x$ is another form of the same root if x is any integer. Likewise, if c is any primitive root of

$$x^{\lambda(p_{r+1}^{a_{r+1}})} \equiv 1 \pmod{p_{r+1}^{a_{r+1}}}, \tag{4}$$

another form of the root is $c + p_{r+1}^{a_{r+1}} y$, where y is any integer. If x and y can be so chosen that

$$h + p_1^{a_1} \cdots p_r^{a_r} x = c + p_{r+1}^{a_{r+1}} y,$$

either member of this equation will be a common primitive root of congruences (3) and (4); that is, a common primitive root of the two congruences may always be obtained provided that the equation

$$p_1^{a_1} \cdots p_r^{a_r} x - p_{r+1}^{a_{r+1}} y = c - h$$

has always a solution in which x and y are integers. But since the coefficients of x and y are relatively prime, the equation has always a solution in integers.

Now let g be the common primitive λ-root of congruences (3) and (4) and write

$$g^\alpha \equiv 1 \pmod{n},$$

where α is to be the smallest integer for which the congruence is true. Since g is a primitive λ-root of (3), α is a multiple of $\lambda(p_1^{a_1} \cdots p_r^{a_r})$. In the same way it is a multiple of $\lambda(p_{r+1}^{a_{r+1}})$. But $\lambda(n)$ is the lowest common multiple of $\lambda(p_1^{a_1} \cdots p_r^{a_r})$ and $\lambda(p_{r+1}^{a_{r+1}})$; therefore α is a multiple of $\lambda(n)$, and hence $\alpha = \lambda(n)$ in view of the analogue of Fermat's theorem already demonstrated; for $g = h + p_1^{a_1} \cdots p_r^{a_r} x = c + p_{r+1}^{a_{r+1}} y$ is evidently prime to n. Therefore g is a primitive λ-root of

$$x^{\lambda(n)} \equiv 1 \pmod{n}.$$

The theorem announced follows by simple induction.

There is nothing in the preceding argument to indicate that the primitive λ-roots of (2) are all in a single set obtained by taking the powers of some root g; in fact this is not even usually

so when n contains more than one prime factor. By taking powers of a primitive root g a set of primitive roots is obtained which evidently is identical with the set obtained by taking powers of any other root belonging to the set. We may say then that the set thus obtained is the set belonging to g. Then

III. *If* $\lambda(n) > 2$, *the product of the primitive roots in the set belonging to any* g *is congruent to* $1 \pmod{n}$.

These primitive roots are

$$g,\ g^{c_1},\ g^{c_2},\ \cdots,\ g^{c_\nu},$$

where $1, c_1, c_2, \cdots, c_\nu$ are the integers less than $\lambda(n)$ and prime to it. If any one of these is c, another is $\lambda(n) - c$ when $\lambda(n) > 2$. Hence

$$1 + c_1 + c_2 + \cdots + c_\nu \equiv 0 \quad [\bmod \lambda(n)].$$

Therefore

$$g^{1+c_1+c_2+\cdots+c_\nu} \equiv 1 \pmod{n}.$$

Hence the theorem.

COROLLARY. *The product of all the primitive* λ*-roots of* $x^{\lambda(n)} \equiv 1 \pmod{n}$ *is congruent to* $1 \pmod{n}$ *when* $\lambda(n) > 2$.

When n is given it is of course a very easy matter to find $\lambda(n)$. But the inverse problem, to find every x such that

$$\lambda(x) = a \quad \text{or} \quad \lambda^{-1}(a) = x, \tag{5}$$

is more difficult. We construct a method for solving this problem.

IV. *If* x_1 *is the largest value of* x *satisfying* (5), *any other solution* x_2 *is a factor of* x_1.

Suppose that p^α is the highest power of any prime p such that $\lambda(p^\alpha)$ is a factor of a. Then evidently p^α is a factor of x_1; but no higher power of p is a factor of x_2, and therefore the theorem follows. Hence the following method for solving the problem in consideration:

Obtain the largest solution x_1 *of* (2); *examine every divisor* d *of* x_1 *and retain those* d*'s for which* $\lambda(d) = a$. *These are all the solutions of* (5).

To make the rule effective we require a means of computing x_1. It is evident that the following method leads to the desired result: Separate a into its prime factors and find the highest power p^α of each prime p contained in a such that $\lambda(p^\alpha)$ is equal to or is a factor of a. Suppose that the following prime powers are found: $p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_i^{\alpha_i}$. Then write out all the divisors of

a and take every prime q such that $q-1$ is equal to any one of these divisors, but q is not eqnal to any p; and say we have q_1, $q_2, \cdots, q_k$. Then

$$x_1 = p_1^{a_1} p_2^{a_2} \cdots p_i^{a_i} q_1 q_2 \cdots q_k. \tag{6}$$

V. COROLLARY. *If y_1 and x_1 respectively are the largest solutions of the equations*

$$\lambda^{-1}(ma) = y, \quad \lambda^{-1}(a) = x, \tag{5a}$$

where m is any integer > 1, then $y_1 > x_1$.

(It is to be observed that in equations (5) and (5a), a and ma are assumed to be numbers such that each equation has at least one solution.)

By aid of theorem IV and the rule based on it I have constructed the following table containing every n for each $\lambda(n) > 1$ and $\leqq 24$. It is interesting to notice that $\lambda(x) = 12$ has 84 solutions.

$\lambda(n)$	n
2	3, 4, 6, 8, 12, 24.
4	5, 10, 15, 16, 20, 30, 40, 48, 60, 80, 120, 240.
6	7, 9, 14, 18, 21, 28, 36, 42, 56, 63, 72, 84, 136, 168, 252, 504.
8	32, 96, 160, 480.
10	11, 22, 33, 44, 66, 88, 132, 264.
12	13, 26, 35, 39, 45, 52, 65, 70, 78, 90, 91, 104, 105, 112, 117, 130, 140, 144, 156, 180, 182, 195, 208, 210, 234, 260, 273, 280, 312, 315, 336, 360, 364, 390, 420, 455, 468, 520, 546, 560, 585, 624, 630, 720, 728, 780, 819, 840, 910, 936, 1008, 1040, 1092, 1170, 1260, 1365, 1456, 1560, 1638, 1680, 1820, 1872, 2184, 2340, 2520, 2730, 3120, 3276, 3640, 4095, 4368, 4680, 5040, 5460, 6552, 7280, 8190, 9360, 10920, 13104, 16380, 21840, 32760, 65520.
16	17, 34, 51, 64, 68, 85, 102, 136, 170, 192, 204, 255, 272, 320, 340, 408, 510, 544, 680, 816, 960, 1020, 1088, 1360, 1632, 2040, 2720, 3264, 4080, 5440, 8160, 16320.
18	19, 27, 38, 54, 57, 76, 108, 114, 133, 152, 171, 189, 216, 228, 266, 342, 378, 399, 456, 513, 532, 684, 756, 798, 1026, 1064, 1197, 1368, 1512, 1596, 2052, 2394, 3192, 3591, 4104, 4788, 7182, 9576, 14364, 28728.
20	25, 50, 55, 75, 100, 110, 150, 165, 176, 200, 220, 275, 300, 330, 400, 440, 550, 600, 660, 825, 880, 1100, 1200, 1320, 1650, 2200, 2640, 3300, 4400, 6600, 13200.
22	23, 46, 69, 92, 138, 184, 276, 552.
24	224, 288, 416, 672, 1120, 1248, 1440, 2016, 2080, 2912, 3360, 3744, 6240, 8736, 10080, 14560, 18720, 26208, 43680, 131040.

VI. *Let a be that divisor of α for which $\lambda^{-1}(a) = x$ has a greatest solution x_1 greater than such a solution when for a any other divisor of α is taken. Then x_1 is the largest divisor of $z^\alpha - 1$ for every z prime to the divisor.*

That x_1 divides $z^\alpha - 1$ follows from Theorem I. Let y_1 be any number greater than x_1. Then in view of the conditions in the proposition $\lambda(y_1)$ is not a divisor of α. Hence, from the foregoing theory of primitive roots, it follows that there is some number z such that $z^\alpha - 1$ is not divisible by y_1. Hence the theorem. (From V. it is seen that $a = \alpha$ when $\lambda^{-1}(a) = x$ has a solution.)

In a previous paper * I tabulated a function $M(a)$ for possible values of a up to $a = 150$. A reference to the definition of $M(a)$ there given will show that $2M(a)$ is identical with our present x_1, the largest solution of $\lambda^{-1}(a) = x$, provided this equation has a solution. That table will therefore serve for determining x_1 for $a \leqq 150$. Thus it is seen that the largest divisor of $z^{144} - 1$ for every z which is prime to the divisor is 685,933,859,520. Further, the table for $M(a)$ may also be used in continuing the table of the present paper.

Professor J. H. Jeans † and more recently Mr. E. B. Escott ‡ have discussed the converse of Fermat's theorem, showing that the relation

$$e^{n-1} \equiv 1 \pmod{n}, \tag{7}$$

which is always true, when n is prime, for any value of e prime to n, is for any particular value of e true for values of n which are not prime. This result will be extended by proving the theorem that *there are values of composite n for which relation* (7) *is true when e is any number prime to n.* In view of the foregoing theory of the congruence

$$e^{\lambda(n)} \equiv 1 \pmod{n},$$

it is evidently necessary and sufficient for this result that n has the property

$$n - 1 \equiv 0 \quad [\text{mod } \lambda(n)]. \tag{8}$$

When $n > 2$, $\lambda(n)$ is even; and therefore (8) can be true for composite n only when n is odd. Further, since $\lambda(n)$ is prime

* BULLETIN, ser. 2, vol. 15, no. 5 (February, 1909), p. 222.
† *Messenger of Mathematics*, vol. 27, p. 174.
‡ *Messenger of Mathematics*, vol. 36, p. 175.

to n it follows that n contains no repeated prime factor; and hence *n is a product of odd primes no one of which is repeated.*

That n is not the product of two odd primes is easily shown. Suppose $n = p_1 p_2$, $p_2 > p_1$. Then

$$\frac{p_1 p_2 - 1}{p_2 - 1} = p_1 + \frac{p_1 - 1}{p_2 - 1} \neq \text{integer}.$$

But $\lambda(n)$ contains the factor $p_2 - 1$ and is therefore not a divisor of $n - 1 = p_1 p_2 - 1$.

On the other hand it is easy to find values of $n = p_1 p_2 p_3$, satisfying relation (8). It is necessary and sufficient that

$$\frac{p_i p_j p_k - 1}{p_i - 1} \equiv \frac{p_j p_k (p_i - 1) + p_j p_k - 1}{p_i - 1} = \text{integer},$$
$$(i, j, k = 1, 2, 3 \text{ in some order});$$

that is, that $(p_j p_k - 1)/(p_i - 1) = \text{integer}$. The following values of n have been found by inspection using this relation:

$$3 \cdot 11 \cdot 17, \quad 5 \cdot 13 \cdot 17, \quad 7 \cdot 13 \cdot 31, \quad 7 \cdot 31 \cdot 73.$$

By a similar method one may seek values of n for which n is the product of four or more primes; but the work will not be carried out here.

An example given by Lucas,* illustrating the failure of the converse of Fermat's theorem, belongs to a different class of exceptions. He shows that

$$2^{n-1} \equiv 1 \pmod{n}, \text{ when } n = 73 \cdot 37.$$

Here $\lambda(n) = 72$ while $n - 1 = 36 \cdot 75$; or $n - 1$ is a multiple of $\frac{1}{2}\lambda(n)$. Then we can easily find other values than 2 and its powers for which the preceding congruence is true. In fact every number prime to n belongs to some index which is a divisor of $\lambda(n)$. But every divisor of $\lambda(n) = 72$ except 8, 24, 72 is a divisor of $n - 1$. Hence the congruence $a^{n-1} \equiv 1 \pmod{n}$ is true for any integer a prime to n and not belonging to the index 8, 24, or 72 (mod n). Most of the examples given by Escott in the paper already referred to are similar to this one. These, however, are not so interesting as those for which congruence (7) is true for any e prime to n.

* Théorie des nombres, p. 422.

BAIRE'S LEÇONS D'ANALYSE.

Leçons sur les Théories générales de l'Analyse. Par RENÉ BAIRE. Tome I: *Principes fondamentaux, Variables réelles.* 1907. 8vo. 17 figures, x + 232 pp. 8 fr. Tome II: *Fonctions analytiques, Équations différentielles, Applications géométriques, Fonctions elliptiques.* 1908. 8vo. 35 figures, x + 347 pp. 12 fr. Paris, Gauthier-Villars.

THE prefaces of these books express in vigorous and convincing language the author's beliefs and plans. Baire has abundantly demonstrated his right to strong opinions; what he thinks regarding comparatively elementary instruction is not to be despised. As a specimen, I quote the following to avoid loss of force in translation: Rigeur et simplicité ne sont nullement inconciliables, si l'on prend nettement le parti de faire pénétrer, dans l'enseignement des principes fondamentaux, certaines idées qui ont été acquises à la Science dans l'étude de questions d'ordre plus élevé. Pour en prendre un exemple frappant, la notion de bornes supérieures et inférieures d'un ensemble, qui commence seulement à être vulgarisée."

This bugle call to the standard of rigor may affright some to whom rigor and difficulty seem synonymous: precisely to such persons Baire's treatise will be a revelation — it *is* simple. That it is also reasonably accurate, Baire's name and the preceding quotation guarantee; indeed one's expectation outruns the author's intention, and one notes the careful avoidance of difficult questions far more than any tendency to finesse.

A very special interest attaches to the introductory work and to the treatment of the foundations of the subject, both because the presentation is somewhat novel, and because the main body of the work is strictly limited to rather usual topics which give rise to little comment.

The first chapter is a treatment of the fundamental concepts of irrational numbers, sets of points, limits, and continuity. As noted in the preface, this chapter is substantially a reproduction of Baire's Théorie des nombres irrationels, des limites et de la continuité, published in 1905 by Vuibert et Nony. As a whole it is clear, exact, and elegant, and may well serve any student as an introduction to this subject.

A consistent treatment of irrationals by the Dedekind cut process is followed by brief treatments of the bounds of an assem-

blage and of the limits of sequences. In the latter topics $\pm\infty$ are recognized as bounds, or as limits "in the extended sense"; whether such infinite limits are excluded or included either is stated or is determinable from the context, but the student might go astray occasionally, for example, in section 20, page 13.

In defining continuity, the sequence notion is used on page 21, but it is shown later, on page 46, that this definition coincides with the usual ε definition. The definition of uniform continuity is phrased in ε form immediately. These definitions are given for several variables, and essentially for any point set; they are then used to establish the four fundamental operations of algebra by means of the fundamental principle of extension of definition (page 28) from the rational to the real numbers.

There follow interesting treatments of concrete quantities, of relations of geometry to algebra, of sets of points, and of the properties of continuous functions. Among other fundamental topics, I shall mention the proof that the two definitions of continuity are equivalent, and the theorem on inverse functions (page 51). The latter, together with the fundamental theorem of extension, is used to establish such functions as x^y, $\log x$, and so on. The chapter ends with a very brief, but satisfactory, treatment of elementary theorems on series of constant terms.

The fundamental principles of the differential and integral calculus are welded in Chapter II into a single link in the mathematical chain; in fact, the traditional separation of those topics is characterized in the preface of the second volume as "superannuated." The sums which lead to the integral are introduced on page 74 via the law of the mean, which, together with other usual properties of derivatives, has been carefully proved. Though this does "lead us to study such sums a priori," the device lacks the ring of naturalness — the weld is obviously artificial. Still, the process leads directly to the fundamental properties of the integral without resort to geometry; the most important of these properties are derived here with characteristic brevity and clearness.

The remainder of the chapter contains rather usual theorems on elementary, improper, and parametric integrals, implicit functions and functional dependence, derivatives and differentials of higher order, differential equations and total differentials. Indeed, I need only mention one or two points to show that the author was looking rather toward broad presentation under

somewhat easy restrictions than toward the extreme niceties. Thus the original definition of integral (page 74) is restricted to continuous functions (extended later in the case of improper integrals); the theorem on implicit functions assumes the existence and continuity of all the first derivatives; and so on.

There are even one or two cases of what might seem carelessness in another, but which are doubtless rather evidences of Baire's desire to avoid complicating detail. For example, the hypothesis that dy/dx or else dx/dy should exist and be different from zero is suppressed on page 65. On page 72, section 84, the condition that f'_x, etc., be continuous, which is necessary, *for this proof*, is omitted. On page 117, n is used in two senses, for it is necessary to assume, *for this proof*, that the derivatives of order $n+1$ are continuous. The hypothesis that f'_x, f''_{xx}, etc., be continuous is omitted on page 121 in the theorem on d^2u. Finally, in section 94, page 81, a proof of the formula for change of variable in an integral is given in new form; it is quite satisfactory, but to the author's hypothesis that $x = \phi(t)$ possess a continuous derivative may well be added the provision that t_0 and t_1 be so chosen that x remains in the interval $x_0 \leqq x \leqq x_1$.

The author's opinions are on the whole against the use of higher differentials. Even the one advantage recognized (preface, page vi) — that of writing relations valid for any later choice of independent-variables — is by no means impossible with the derivative notations, and the fact mentioned in lines 18–19 of page 125 seems to rob this advantage of its force.

Among the most interesting and novel features of the whole work are the very elegant presentations of the ideas of length, area, and volume in Chapter III. A precise treatment of length is followed by an axiomatic development of area — which is not used in the original presentation of integration. While the ideas are not new, the presentations are so; they will be found interesting even to the initiated. A still simpler presentation seems possible, however, in the case of the area under a continuous curve $y = f(x) \geqq 0$. For let us assume the following axioms:

(1) The area of any rectangle is the product of its base and its height.

(2) The area of the figure formed by placing a finite number of rectangles in juxtaposition is the sum of the areas of those rectangles.

(3) The area of any region, if it exists, is a unique number.

(4) The area of any part of a region is less than or at most equal to the area of the whole region.

With these axioms, it follows immediately that the area between $x = a$, $x = b$, $y = 0$, $y = f(x)$ is necessarily $\int f(x)\,dx$, whenever that integral exists in Riemann's sense. Such a presentation, while admittedly not quite so far reaching as that of the text, would be wholly in keeping with the spirit of the work as a whole, in avoiding extreme cases.

Before passing to the question of volume, the double integral is defined, and its properties are discussed. The proof of the formula for change of variables is mentioned in the preface, and deserves notice here. The proof is given first for linear transformations, and is extended to the general case by use of the law of the mean. It is evident that Baire considered this presentation very carefully; the only objection appears to be its length, which seems out of proportion to some other parts in view of the fact that other proofs — artificial, to be sure — are available.

The treatment of improper double integrals (pages 177–179) is so brief that it may be misleading; the impression that circles may be used exclusively is not strictly justified by the text, but that impression will doubtless be received by a superficial reader.

The discussion of volume is exactly similar to that of area. It is followed by a treatment of triple integrals and a very extended treatment of the area of a surface. Especially the latter will be found interesting, but it is clearly out of proportion to the rest of the work. An excellent presentation of line and surface integrals, together with the theorems of Green and of Stokes, concludes this chapter and the first volume.

Volume II opens with a rather brief treatment of functions of complex variables. The presentation is general rather than an exposition of any one school; the demonstrations are conducted by methods of real variables whenever desired; emphasis is laid on the developability of any function which has a derivative; and only theorems of unquestionable prominence are given. Functions of several complex variables receive considerable attention; theorems are proved for several variables when convenient; and the results are used in proving such theorems as that for differentiating under the integral sign.

The most interesting portion of this chapter is the section on infinite series, which includes the theory of real series, since only series of constant terms were treated in Volume I. Baire makes extensive use of what he calls "normally convergent"

series, i. e., series which satisfy the Weierstrass test for uniform convergence. It is shown that any uniformly convergent series may be transformed into a normally convergent series by properly grouping consecutive terms, and the usual forms of statement are deduced from the results for normally convergent series. The new phrase is certainly convenient; its use in this book abundantly justifies its introduction.

Chapter V contains a treatment of differential equations, confined for the most part to elementary methods of actual solution. It includes, however, an existence proof for a set of linear equations (pages 86–89), and a very excellent treatment of characteristics of differential equations of the first order (pages 148–164). Partial differential equations of higher order are barely mentioned.

Chapter VI deals with applications to geometry. Again the work is quite restricted to very usual theorems, the topics considered being the usual elementary theorems regarding such ideas as lines of curvature, asymptotic lines, applicability of surfaces, geodetic lines, and so on. The presentation is far more traditional than in the previous chapters.

The final chapter is entitled elliptic functions, but a considerable portion of it is devoted to infinite products. The treatments are quite elementary and traditional, as in several chapters of this volume. Here, as in the chapter on complex variables, little is said of geometric representation of the Riemann type — another example of the policy of rigid exclusion of all dispensable materials.

On the whole, Baire's work fulfils the promise of his prefaces; the work is simple, restricted to rather fundamental topics, and yet accurate. The manner of presentation is often quite new, and is always clear and effective. There is some lack of uniformity, some loss of proportion, of one topic as compared with another; but this must remain a question of opinion, possibly of tradition, even of prejudice. Thus the foundations are laid wide and strong and as if for eternity. That the superstructure is even less imposing than in many of the older Cours may be disappointing; it would seem reasonable and reassuring, however, to assume that Baire wished to build so as to leave for others not the task of rebuilding from the very ground what he has done here, but rather that of starting where he stops, assured that the foundations he has laid will stand the strain of an enormous superstructure.

E. R. HEDRICK.

INFINITE SERIES.

Lehrbuch der unendlichen Reihen. Von Dr. Niels Nielsen. Vorlesungen gehalten an der Universität Kopenhagen. Leipzig, Teubner, 1909. viii + 287 pp. Price 12 M.

This excellent elementary textbook shows the careful working over which is necessary to make a course of lectures fit the requirements of the classroom. From the beginning to the end the author constantly has in mind the students before him. The foundations do not presume a partial structure already in the students' minds, but start from the bare fundamentals all must possess to understand the course at all. For this reason the work is divided into three parts: theory of sequences, series with constant terms, series with variable terms. We may say that the first part defines what a series must mean, and what it can give us; the second part discusses the management of the particular value of a series for a given value of the variable; the third part discusses the sweep of values due to different values of the variable. The development proceeds leisurely and is well illustrated with examples. The references are sufficiently numerous to incite the student to follow up the subject in original papers, but not so exhaustive as to overwhelm him.

The first part contains six chapters, in order: rational numbers, irrational numbers, real sequences, complex sequences, applications to elementary transcendental functions, doubly infinite sequences. The conception of rational numbers and their combinations under the four processes of arithmetic is the beginning. From this is developed immediately the idea of rational sequence, and limit. It is then shown that any periodic decimal fraction represents a rational number, and that non-periodic decimals represent limits but not rational numbers. An irrational number a is defined to be the non-rational limit of two approximation series h_n, l_n, such that

$$h_n > a > l_n, \quad \lim (h_n - a) = 0, \quad \lim (a - l_n) = 0.$$

The irrational ω, $h_n > \omega > l_n$, is greater than the irrational ω', $h'_n > \omega' > l'_n$, if for a certain n, $l_n > h'_n$; and ω is less than ω', if for a certain n, $h_n < l'_n$. In any other case, for each n, $h_n \geqq l'_n$ and $h'_n \geqq l_n$ and $\omega = \omega'$. The four fundamental operations are defined for irrationals by defining the limits of $h_n \pm h'_n$, $l_n \pm l'_n$,

or $h_n\ h'_n$, $l_n\ l'_n$, $h_n : h'_n$, $l_n : l'_n$, respectively. With these definitions it is shown that every irrational has an incommensurable decimal expression, and that irrational sequences lead to no new varieties of number. Thus the rational sequence really closes the number system. The development followed here is a special case of that of G. Cantor.*

Sequences of real numbers are classified as convergent or fundamental series; divergent, which are either proper or improper according as the sequence from a certain term on retains the same sign or not; and oscillating. The limits of the sum, difference, product, and quotient of two fundamental series are shown to be the sum, difference, product, and quotient, respectively, of the two limits. The general test of convergence is proved in these terms: †

The necessary and sufficient condition for the convergence of the sequence $a_0, a_1, \cdots, a_n, \cdots$ *is that for any preassigned arbitrary small positive number* ϵ *there exists a positive integer* N, *such that for* $n \geqq N$

$$| a_{n+p} - a_n | < \epsilon$$

for any positive integer p.

The author now introduces the conceptions of monotone sequence, of limit points, of upper and lower boundaries, of superior and inferior limits. He closes the chapter with a theorem of Abel: ‡

If there is given a monotonic non-increasing sequence ϵ_n, *and if from a given sequence* $\cdots, a_n, \cdots$ *we form a new sequence* $\cdots, s_n, \cdots$, *where* $a_p = s_p - s_{p-1}$, *then for every* n

$$\epsilon_1 \cdot g_n \leqq a_1 \epsilon_1 + \cdots + a_n \epsilon_n \leqq \epsilon_1 \cdot G_n,$$

where G_n and g_n are respectively the upper and lower boundaries of the sequence s_n.

In the next chapter, on complex sequences, the complex number is treated as a couple (a, b). The usual formulas of combination are developed, and the convergent complex sequence is defined to be one such that the sequence of the moduli of its terms is convergent. A theorem of Jensen,§ which is a generalization of one of Cauchy, is given:

* *Math. Annalen*, vol. 5 (1875), p. 128.

† Du Bois-Reymond: Allgemeine Funktionentheorie, vol. 1, p. 6, 260.

‡ *Crelle*, vol. 1 (1826), p. 314.

§ *Tidsskrift for Mathematik*, ser 5, vol. 2. (1884), pp. 81–84. *Comptes Rendus*, vol. 106 (1888), pp. 833–836.

If the sequence $\cdots, a_n, \cdots$ *satisfies the two conditions*

$$(1) \qquad \lim_{n=\infty} |a_n| = \infty,$$

and for every n

$$(2) \quad |a_1| + |a_2 - a_1| + |a_3 - a_2| + \cdots + |a_n - a_{n-1}| < B.|a_n|,$$

where B *is a fixed determinate number independent of* n*; if moreover the fundamental series* $\phi_1, \phi_2, \cdots \phi_n, \cdots$ *has the finite limit* $\Phi = \lim_{n=\infty} \phi_n$*, then we also have*

$$\Phi = \lim_{n=\infty} \left(\frac{a_1 \phi_1 + (a_2 - a_1)\phi_2 + \cdots + (a_n - a_{n-1})\phi_n}{a_n} \right).$$

From this we may deduce the formula

$$\lim_{n=\infty} \left(\frac{b_n}{a_n} \right) = \lim_{n=\infty} \left(\frac{b_n - b_{n-1}}{a_n - a_{n-1}} \right),$$

where $b_n = a_1 \phi_1 + (a_2 - a_1)\phi_2 + \cdots + (a_n - a_{n-1})\phi_n$.

For $a_n = n$, we have a theorem of Cauchy :*

If in the sequence $\cdots, b_n, \cdots$ $\lim_{n=\infty} (b_n - b_{n-1})$ *is finite and determinate, then*

$$\lim_{n=\infty} \left(\frac{1}{n} b_n \right) = \lim_{n=\infty} (b_n - b_{n-1}).$$

In the second part, which treats of real series, a series is defined to be a sequence $\cdots, s_n, \cdots$ of sums of terms $u_1, \cdots, u_n, \cdots$. Convergence of a series is distinguished from summability. If the new sequence $S_n = (s_1 + s_2 + \cdots + s_n)/n$ is formed and if it is a convergent sequence, then the series $u_1, u_2, \cdots, u_n$, which gives $\cdots, s_n, \cdots$ is said to be summable. A generalization of this is also suggested, through the Jensen theorem mentioned above. Convergence which is independent of the order of the terms is unconditional; otherwise it is conditional. Absolute convergence exists when the series of absolute values of the terms converges. It is shown that unconditional convergence and absolute convergence imply each other.

Certain cases of convergence are now considered. These are: the well-known theorem of Leibniz on alternating series;

* Analyse algébrique, 1821, p. 54.

comparison series; a theorem of Pringsheim to the effect that *if* $\cdots, a_n, \cdots$ *is monotone increasing and divergent, then the series whose terms are*

$$u_n = \frac{a_{n+1} - a_n}{a_{n+1} \cdot a_n^\rho}$$

is convergent for every positive ρ; logarithmic tests; a theorem of Jensen: *

If $\lim_{n=\infty} |a_n| = \infty$, *and* $|a_1| + |a_2 - a_1| + \cdots + |a_n - a_{n-1}| < B \cdot |a_n|$, *but otherwise the values of* a_n *are quite arbitrary, then*

$$\sum_{n=1}^{\infty} \frac{a_n - a_{n-1}}{a_n}$$

is divergent; several conclusions are drawn from this, embodying theorems of Dini; two theorems ascribed to both Du Bois-Reymond † and Dedekind ‡:

1. *If* $\cdots, a_n, \cdots$ *and* $\cdots, b_n, \cdots$ *are two infinite series of arbitrary complex numbers which satisfy only the conditions that both*

$$\sum_{n=0} a_n \text{ and } \sum_{n=0}^{\infty} |b_n - b_{n+1}|$$

converge, then $\sum a_n b_n$ *converges.*

2. *If* $\cdots, a_n, \cdots$ *and* $\cdots, b_n, \cdots$ *satisfy only the conditions* $\lim_{n=\infty} b_n = 0$, $\sum |b_n - b_{n+1}|$ *converges, and* $\sum a_n$ *oscillates between finite boundaries, then* $\sum a_n b_n$ *converges.*

The chapter following is devoted to elementary tests of convergence. This seems to be a purely artificial separation of the subject treated in this chapter and the preceding. The tests given are (1) Kummer's: §

The series of positive terms $\sum u_n$ *is convergent if it is possible to determine a series of positive terms* $\cdots, \phi_n, \cdots$ *such that for* $n \geqq N$

$$\phi_n \cdot \frac{u_n}{u_{n+1}} - \phi_{n+1} \geqq \alpha > 0.$$

Likewise divergent if

$$\phi_n \cdot \frac{u_n}{u_{n+1}} - \phi_{n+1} < 0.$$

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* *Tidsskrift for Mathematik* (1884), ser. 5, vol. 2, p. 85.

† Neue Lehrsätze über die Summen unendlicher Reihen, p. 10.

‡ Dirichlet, Vorlesungen über Zahlentheorie, § 101, 3 Aufl., 1879.

§ *Crelle*, vol. 13 (1835), p. 172.

(2) Cauchy's: *

The series of positive terms $\sum u_n$ *is convergent or divergent according as*

$$\lim_{n=\infty} \inf \frac{u_n}{u_{n+1}} > 1 \text{ or } \lim_{n=\infty} \sup \frac{u_n}{u_{n+1}} < 1.$$

(3) Duhamel's: †

If for the series of positive terms $\sum u_n$ *we have a development of the form*

$$\frac{u_n}{u_{n+1}} = 1 + \frac{1+\alpha}{n} + \cdots,$$

the series converges or diverges according as $\alpha > 0$ or $\alpha < 0$.

(4) Raabe's: ‡

If for the series of positive terms $\sum u_n$ *we have*

$$\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{a}{n^{1+\delta}} + \cdots,$$

where a *is finite and* δ *is an arbitrary small but assignable positive number, then* $\sum u_n$ *diverges.*

(5) The logarithmic test.

Following these is a chapter on the multiplication of series, which is given a somewhat original treatment.

After an excellent chapter on transformation of series for more rapid numerical computation, the question of convergence is again taken up. The preceding theorems are widened and generalized and the discussion is led to the general criteria of Pringsheim. From his criteria of the first and second kinds are shown to follow all the more elementary criteria. It seems unnecessary to trace the development further. It is evidently along teachable lines, and will give the student a clear understanding of the question of convergence. This part is closed with a treatment of infinite products, infinite continued factors, and double series.

The last part treats of series of variable terms. In the first chapter the idea of point set (Menge), is developed, with its usual accompanying terms. In the second chapter the idea of

* Analyse algébrique, p. 134.

† *Journ. de Math.*, vol. 4 (1839), pp. 214–221.

‡ *Journ. von Ettinghausen und Baumgartner*, vol. 10 (1832); *Crelle*, vol. 11, pp. 309–311. *Journ. de Math.*, vol. 6 (1841), pp. 85–88.

function of one or two real variables, and of a complex variable is developed. The function is handled as a point set of functional values. The general principle of convergence is stated in these terms:

The necessary and sufficient condition for the existence of $f(a-0)$, *and thus for the existence of a limit* A *for the function* $f(x)$ *as* x *approaches* a, *is that it be possible to find for any preassigned arbitrary small number* $\epsilon > 0$ *a number* $\delta > 0$ *such that if* $a - \delta \leqq x \leqq a$, $a - \delta \leqq y \leqq a$, *then* $|f(x) - f(y)| < \epsilon$.

A similar condition is stated for a function of a complex variable. The definition of, and conditions for, continuity follow. It is shown that any rational function of continuous functions of x is continuous, save for those values which make the resulting denominator vanish. Also if $y = f(x)$ is continuous for $x = a$, and $z = \phi(y)$ is continuous for $y = b = f(a)$ then $z = F(x) = \phi[f(x)]$ is continuous for $x = a$. The existence of maxima and minima is then proved as usual.

The next chapter treats of uniform convergence. Seven theorems are given to the effect that

1. *A convergent sequence of continuous functions of* x *which is uniformly convergent in a given interval defines a continuous function of* x *over this interval.*

2. *An infinite convergent series is uniformly convergent over an interval if the remainder after the nth term is convergent independently of the values of* x *in the interval.*

3. *An infinite convergent series of continuous functions of* x *which is uniformly convergent over an interval defines a continuous function of* x.

4, 5, 6, 7. These relate to the convergence of infinite products.

To illustrate these theorems the gamma function is briefly treated. The following chapters discuss trigonometric series, power series, Dirichlet series, and faculty series. In the latter chapter Professor Nielsen has incorporated some original work. The 210 problems are very carefully selected to throw light on the developments which they follow. The text as a whole is rather carefully graded as to difficulty and yet is full enough for a first course. Some few errors remain uncorrected but these are easily apparent to the reader.

JAMES BYRNIE SHAW.

THE COLLINEATIONS OF SPACE.

Die Lehre von den geometrischen Verwandtschaften. Dritter Band: die eindeutigen linearen Verwandschaften zwischen Gebilden dritter Stufe. By RUDOLF STURM. Leipzig and Berlin, B. G. Teubner, 1909, viii + 574 pp.

THE third volume of Professor Sturm's treatise has about the same range of discussion for three dimensions as the preceding ones had for one and two respectively,* the size of the volume being explained by the very full treatment given to systems of correlations. This volume is not provided with a separate index nor a preface, but does contain a glossary of 165 technical words not found in the preceding ones, and the table of contents of all four volumes.

The first chapter (pages 1–352) is concerned with the ordinary problems of collineation and correlation in space, being somewhat similar to the treatment found in the corresponding parts of Reye's Geometrie der Lage, but much more extensive. The properties of central and axial perspective, affinity, similarity, and congruence are first brought out to discuss the collineations and polarity defined by linear complexes. All of this leads up to the theorem that a general collineation can be resolved into three axial involutions.

The two kinds of collineation which leave a given quadric invariant, and in particular a space cubic curve are given thirty pages. This could have been materially shortened by a more liberal use of analytic methods,† the development being almost exclusively synthetic. In fact the need of still another treatise, having the same general content as this first chapter, but proceeding algebraically, is keenly felt. It could be regarded as an appendix to the work of Professor Sturm.

Metrical, particularly focal, properties follow. Here a number of results are necessarily stated in algebraic form, but mostly without proofs. A knowledge of projective analytic geometry of quadrics is presupposed. Besides lines of curva-

*See the reviews in volume 15 of the BULLETIN; that of Volume 1 on page 135, and of Volume 2 on page 252.

† As for example, that employed by Wiman; "Ueber die algebraischen Curven von den Geschlechtern $p=4$, 5, and 6, welche eindeutige Transformationen in sich besitzen," *Bihang till Kongl. Svenska Vetenskaps-Akad. Handlingar*, 1895, and particularly, Miss Van Benschoten; "Birational transformations of algebraic curves of genus four," *Amer. Jour. Math.*, vol. 31 (1909), pp. 213-252.

ture and curves of contact with an enveloping developable, geodesics on a quadric are defined and shown to be covariant in the collineations of a system of confocal quadrics. A few general theorems concerning the commutativity of two collineations are followed by a more thorough discussion of groups generated by central harmonic homologies. Every axial involution can be generated in this way. Cyclic collineations and cyclic correlations of degree four are treated at length.

Over sixty pages (287–352) are devoted to the discussion of groups of collineations. The first topic is that of collineations in inscribed tetrahedral position, i. e., such that a tetrahedron exists having the property that the image of each vertex lies in the opposite face. It is shown that the existence of one such tetrahedron insures the existence of ∞^9 of them, three vertices being arbitrary. The article on cyclic collineations with non-planar cycles distinguishes between cycles of odd and of even period, and discusses those of period 4, 5, and 6 at length. These themes have been exhaustively treated in a number of recent dissertations from Breslau and Strassburg.*

In case of period six the principal distinction is between four imaginary invariant points on the one hand, and two real and two imaginary ones on the other. In the article on the general group concept a number of properties are mentioned, but the fundamental question, under what conditions are two collineations commutative, is not touched.†

Besides the regular body groups, the G_{32} defined by six linear complexes mutually in involution, and its linear subgroup of order 16 are treated in detail. The chapter closes with a comprehensive synthetic treatment of the configuration $(12_6, 16_3)$ composed of 12 planes, 12 points and 16 lines, such that each of the 16 lines lies in three of the planes and passes through six points, and every point lies in six planes, defining a desmic

* Particularly H. Küppers, "Kollineationen, durch welche fünf gegebene Punkte des Raumes in dieselben fünf Punkte transformiert werden" (Münster, 1890, 79 pp.); H. Reim, "Wie müssen zwei projektivische Punktfelder aufeinander gelegt werden, damit entsprechend kongruente Polygone cyclisch zusammenfallen?" (Breslau, 1879, 28 pp.); R. Krause, "Ueber senäre cyklische Kollineationen" (Strassburg, 1903, 58 pp.); E. Gässler, "Ueber senär cyklische Korrelationen in der Ebene und im Raume" (Strassburg, 1903, 39 pp.); J. Cordier, "Gruppe von 96 Kollineationen (Strassburg, 1905, 58 pp.).

† The theorems given by Reye regarding two commutative linear substitutions are only partly correct. The theorem of Stéphanos for binary fields can be readily extended to any number of dimensions, but other forms can also appear in space of three dimensions.

system. This system belongs to a group of 2304 collineations and correlations, which contains subgroups of order 1152 and of order 576. The simple group of collineations of order 168 first discovered by Klein, and that of order 360 found by Valentiner are mentioned only in a footnote. Notwithstanding the excellent discussion in the present treatise and the many valuable memoirs mentioned in it, there is still much important work to be done before the theory of linear and birational transformations can be fully translated into the language of substitutions.

The second chapter (pages 353–574) is concerned with linear systems of curves and surfaces, singular collineations, problems in enumerative geometry and linear systems of linear transformations. It begins with a system of plane curves having one degree of freedom, then two, etc., each successive case being generated by the fan formation from the preceding one. While this treatment is an excellent one, it could be much improved or at least its results made more useful for others by the addition of an algebraic formulation of both processes and results. The section on the generation of curves and surfaces by means of projective pencils follows Cremona somewhat, but is more readable and much more extensive. The treatment of the important subject of restricted systems is more systematic but less extensive than that found in Salmon's Algebra, the only other book containing it. The theory of poles and polars is essentially that of Cremona, but here again, is much longer. Incidentally a generous synthetic treatment of covariant curves and surfaces including the Hessian, the Steinerian, the Cayleyan, and their Jacobian is introduced.

The long sections on singular correlations and collineations begin with the adjustment of two correlative spaces into a polar space. As first singular form appears the polarity with regard to a cone, or its dual, a conic, and a pair of planes or a pair of points. If now the superposed position of the two spaces be dissolved, and with it the involutorial correspondence, we have the three singular correlations, called by Hirst the central, the planar, and the axial. A well-known special case of the last named is the reciprocity in a special linear complex. By dualizing one of the spaces we obtain the singular collineations of the first kind, but only two forms, as the first and second are really equivalent. A central homology with invariant 0 furnishes an example of the first kind (ordinary artist's perspective)

and an axial collineation with parameter 0 is an illustration of the second kind.

From this foundation it is now possible to reduce the successive cases to depend on those of singular plane collineations which were treated at length in Volume 2. But complete enumeration is a hopeless task; after allowing for all possible duplications, over 4000 signatures exist, each one deserving some attention.

The results of this investigation are employed to determine the number of conditions imposed by homology, axial involution, affinity, etc., and then in turn the possible singular transformations in these various types. Linear systems of space correlations, of polar spaces and duality in a pencil of linear complexes are then treated, and the singular and degraded cases included among them are determined.

The idea of apolar linear systems of space correlations leads to a number of new configurations, and serves to systematize a number of theorems found by Rosanes, Reye, Voss, Kohn, and others.

Given two correlations C, Γ having the property that one tetrahedron $\alpha'\beta'\gamma'\delta'$ in the space Σ' exists such that its image $A_1B_1C_1D_1$ in Γ is inscribed in its image $ABCD$ in C; in the two spaces Σ, Σ' there are then ∞^9 tetrahedra having this property; by performing the operations in inverse order (the C polar of the Γ polar) we have ∞^3 tetrahedra circumscribed about the given ones. The correlations Γ and C are called apolar, C is said to support Γ and Γ to lie in C.

If C_1, C_2 are two correlations, each of which supports Γ, then every correlation of the pencil determined by them will also support it. Similarly for nets, webs, and in general k-fold systems. Simultaneously Γ may be any correlation of an i-fold system, such that $i + k = 14$. This idea is now applied to a singular correlation, i. e., Γ is now polarity with regard to a cone having its vertex at S, and S' the center in Σ'. The points S, S' are conjugate in C. It now follows that every pencil of correlations contains one which supports a given one.

The same procedure is then carried out for collineations, by dualizing one of the spaces and applying (S, s'), (S', s) connexes as was done in the second volume. Finally a long discussion (57 pages) is given to linear systems of projective configurations which support each other.

Notwithstanding the size of the volume many topics are

treated very briefly, and some knowledge of various special operations is presupposed. A familiarity with the preceding volumes would not be sufficient preparation for the intelligent reading of the present one.

VIRGIL SNYDER.

A SYNOPTIC COURSE FOR TEACHERS.

Elementarmathematik vom höheren Standpunkte aus. Von F. KLEIN. Teil I: *Arithmetik, Algebra, Analysis.* Vorlesung gehalten im Wintersemester 1907–08. 5 + 590 pp. Teil II: *Geometrie.* Vorlesung gehalten im Sommersemester 1908. 6 + 515 pp. Ausgearbeitet von E. HELLINGER. Autogr. Leipzig, in Kommission bei B. G. Teubner, 1908–09.

THE volumes under review contain a course of lectures intended for prospective teachers of mathematics in the secondary schools of Germany. The objects of the course and the reasons for giving it are so well stated in the introduction to the first volume and are of such vital interest in their application to conditions in our own country, that it seems desirable to quote at length.

"In recent years" — thus does Professor Klein begin his first lecture — "a widespread interest has developed among university teachers of mathematics and the natural sciences regarding the proper training of teachers for our secondary schools. This movement is of quite recent date; for a long period previously our universities were concerned exclusively with the higher science without any reference to the needs of the secondary schools and, in fact, without attempting to bring about a connection with secondary mathematics. But what is the result of such a practice? The young student at the outset of his university work is brought face to face with problems that do not serve to remind him of what he has previously studied and naturally he proceeds to forget all of it quickly and thoroughly. On the other hand, if after leaving the university he enters upon his work as a teacher, he is required to give instruction in the established courses in elementary mathematics and, as he is unable without assistance to bring his new work into relation with his advanced mathematics, he soon adopts the old traditional methods and his university studies become merely a more or less pleasant memory which has no influence on his teaching.

"At the present time the attempt is being made to destroy this *double discontinuity* which has certainly been in the interest neither of the secondary schools nor of the university. This is to be done, on the one hand by infusing into our secondary school courses new ideas in keeping with the modern development of our science and of culture in general . . ., on the other hand by giving due consideration in our universities to the needs of the prospective teacher. One of the most important means to this end, it seems to me, is a synoptic course of lectures of the kind I begin today. . . . My purpose will be throughout to exhibit *the mutual connection between the problems of the various branches of mathematics* which is not always sufficiently emphasized in the special courses devoted to them, as well as to emphasize their relations to the problems of elementary mathematics. Thereby I hope to make easier for you what I should like to designate as the real purpose of your university study of mathematics; viz., that *you may be able in a large measure to draw inspiration for your teaching from the great body of knowledge that has here been presented to you.*"

It is true also in this country that university courses in mathematics are generally given without direct reference to the needs of those who expect to teach the subject in our secondary schools. The result is that described by Professor Klein: the mathematics studied in the university has little or no influence on the teaching of elementary mathematics. An increasing number of prospective high school teachers are taking advanced courses and advanced degrees at our universities in preparation for their future work. As a result it seems probable that at no distant date a master's degree at least will be required of candidates for positions on the faculties of our best high schools. This is of course as it should be. But it imposes on our universities the duty of offering these men and women carefully planned courses of study that will give them a thoroughly practical and helpful preparation for their later work. The problems connected herewith have received but little attention hitherto in this country. In Germany, on the other hand, they have been widely discussed during the past decade. These discussions culminated in the report of the German commission of 1907 which contains a comprehensive plan of study for the prospective teacher of mathematics or the natural sciences.*

*Die Tätigkeit der Unterrichtskommission der Gesellschaft Deutscher Naturforscher und Ärzte, Leipzig, 1908, pp. 264–306.

This report is of great interest to us in this country and might well serve as a basis for the discussions of an American commission instructed to consider these questions.

The desirability of such a synoptic course as part of the training of a teacher is so obvious that it seems highly probable that courses of this character will be given at our own universities as soon as they begin seriously to consider the needs of those who are preparing to teach. In considering the volumes under review, which give us in detail what their distinguished author believes should form the content of such a course, it seems desirable, therefore, to keep in mind their possible use as a basis for courses of this kind in America.

From what has been said of the nature of the course, it will already be clear that the prospective teacher is to take this course in his last year at the university. It is assumed that he has previously taken courses in analytic geometry, calculus, differential equations, descriptive and projective geometry, the theory of numbers, the theory of curves and surfaces, and the theory of functions, in addition to courses in mechanics, experimental and theoretical physics, chemistry, philosophy, pedagogy, logic and psychology.* Can we expect one of our students to cover this ground, assuming him to spend one year in graduate work, without making his undergraduate course too one-sided? I think we can. Many of our students now cover the work in analytic geometry and calculus, in chemistry and in experimental physics by the end of their sophomore year. The junior year might then bring one semester each (three hours weekly) of differential equations, solid analytic geometry, the theory of numbers, and descriptive geometry, and a full year course in analytic mechanics. The senior year should then contain a full year course in the theory of functions of a complex variable and a full year course in theoretical physics. The work in philosophy, logic, psychology, and pedagogy may be distributed through the junior and senior years. The year of graduate work might then be devoted to curves and surfaces, projective geometry and a synoptic course of the kind now before us. This outline would seem to leave an adequate amount of time during the four undergraduate years for general cultural courses. Many desirable modifications of this outline will suggest themselves. My present purpose is accomplished if I have shown that without im-

* This list is taken from the report of the German commission referred to above.

pairing his undergraduate course along general cultural lines a student can prepare himself along the lines of the plan laid down by the German commission in the time usually required for the obtaining of a master's degree.

We may now turn to the consideration of the contents of the volumes under review. The limitations of space (and of time) prevent a detailed description. We must content ourselves with a general survey, to which may be added the discussion of one or two features which seem to merit special attention. The first volume is divided into three main parts devoted respectively to "Arithmetik" (i. e., the theory of numbers in a broad sense), algebra, and analysis. In some 80 pages the author considers in order the operations with the natural numbers, the fundamental laws to which these operations are subject, the logical foundations of arithmetic (as we use the term in this country, i. e., the German Rechnen), the technique of numerical calculation with the natural numbers (the thoroughly practical point of view is here emphasized by the detailed description of the mechanism of a calculating machine), the successive extensions of the number concept by the introduction of the negative, fractional, and irrational numbers. There follows a section of some 40 pages on the theory of numbers proper, treating in particular the properties of prime numbers, decimal fractions, continued fractions, Fermat's last theorem, cyclotomic problems, and closing with an elementary proof that the regular inscribed polygon of seven sides cannot be constructed with a ruler and compass. Then follows a section of about 40 pages on ordinary and higher complex numbers, with a more detailed consideration of quaternions. This completes the first main part.

It seems desirable at this point to say a word regarding the author's method throughout this volume. In accordance with his avowed purpose of exhibiting everywhere the mutual relations of the various branches of mathematics we have throughout the first volume a wealth of geometric illustrations. We have geometric interpretations of the fundamental laws of algebra (with a vigorous protest against the practice of using such interpetations as proofs in cases where they do not apply), Klein's beautiful geometric representation of a continued fraction, a geometric formulation of the problem of the pythagorean numbers and its extension to that of Fermat's theorem already mentioned. We have already spoken of the application of

analysis to prove the impossibility of constructing with ruler and compass an inscribed regular heptagon. We are given also the usual geometric representation of complex numbers. Finally the multiplication of quaternions is interpreted as rotation and stretching in space. True to his other avowed object, the author also gives at the end of the discussion of each topic what he conceives to be its relation to elementary mathematics and its bearing on the problems confronting the teacher. In the latter particular the method of the first volume differs essentially from that of the second, where all matter relating to the teaching of the elementary branches is reserved for the end and is discussed at length in an appendix. This arrangement is made desirable, to some extent, by the sequence of topics adopted. But the reviewer is inclined to believe that the plan of the first volume is the more effective. Finally, in this connection we must not fail to call attention to the prominent place given throughout every discussion to the history of the subject under consideration. The historical setting is made an essential feature of the presentation and contributes greatly to a clear understanding and the stimulating effect of the whole.

In order to prepare his hearers further in this direction, the author here interposes a general survey of the modern development of mathematics. It is a mere sketch, but it is a masterpiece. In some twenty pages he calls attention first to two fundamental tendencies in the growth of our science. The first tendency has for its object the development of a given branch of mathematics for its own sake and with its own methods. If this tendency alone obtained, mathematics would appear as a group of distinct theories which may show here and there incidental points of contact, but which have no organic unity. The second tendency on the other hand has for its object precisely to wipe out the boundaries between the various so-called branches, and conceives of mathematics as a unified whole, in which the results and methods of every branch are common property of the whole. From this point of view the two fundamental divisions of analysis and geometry become fused into a single whole in which every theorem of either may be regarded as a theorem of the other. It seems to us that the author might have been even more emphatic on this point than he is, a subject to which we will return presently in the discussion of his treatment of the modern developments of the foundations of the science. After calling attention to these two tendencies, the

author sketches briefly the history of mathematics from ancient to modern times with reference to them. This discussion, if printed, would hardly fill more than half a dozen ordinary octavo pages; nevertheless, the author is able to give a remarkably vivid picture of the successive stages in the development of mathematics.

After this interlude, he takes up the second main part of his lectures. In his discussion of algebra he confines himself exclusively to the theory of equations, in particular to the use of graphical and in general of geometric methods in this theory. The first part of the discussion is devoted to the description of certain graphical methods for the solution of real equations with real roots. By the use of the principle of duality two methods of representation are obtained, according as the coordinates are interpreted as point or as line coordinates. This discussion contains much of interest also to the student of analytic geometry and forms an excellent example of the interplay of two branches of mathematics when applied to a special problem. The second part of the treatment of the theory of equations is devoted to the consideration of the equation from the point of view of the theory of functions of a complex variable, the representation by means of conformal mapping on two spheres occupying the central position.

The third main part of the volume, that on analysis, begins with a discussion of the logarithm. After sketching the origin and development of the theory in historical sequence which brings out clearly why it is that the irrational number e should appear as the base of the system of "natural" logarithms, professor Klein concerns himself in detail with the difficulties attendant on the introduction of logarithms in secondary instruction. This is followed by an exposition of how he would like to see this subject treated in elementary courses. His plan contemplates the definition of the logarithm of a as the area between the hyperbola $xy = 1$, the x-axis, the ordinate $x = 1$ and the ordinate $x = a$; *i. e.*, by the relation

$$\log x = \int_1^x \frac{dx}{x}.$$

Whatever may be said of the practicability of this procedure in Germany, there can be little doubt that it is utterly impracticable in this country, at least with the present organization of

our secondary school curricula. With us, logarithms are needed first as an aid in numerical computation in our courses in trigonometry. At the time when it thus becomes necessary to introduce logarithms, the pupil is not even familiar with the hyperbola, to say nothing of his total ignorance of the notion of an integral. However much we may be in sympathy with the movement looking toward an early introduction of the notion of function, graphical representation, and of the elements of the infinitesimal calculus, it appears to the reviewer quite impossible to make these concepts all familiar to a pupil by the time it first becomes desirable to consider logarithms. Judging by a recent German review, conditions in that country do not seem to be very different from our own in this regard.* The reviewer, moreover, is of the opinion that nothing is gained by an attempt to introduce the natural logarithms before the regular course in the calculus. As to the best method of introducing logarithms, when they become desirable as an aid to computation, opinions may differ. The chief pedagogical difficulty at this time is to make clear to the pupil that corresponding to every positive real number there is a logarithm to a given positive base, say 10. The reviewer has found that the following plan works well in practice: Supposing the base to be a, the successive powers

$$(1) \qquad \cdots, a^{-3}, a^{-2}, a^{-1}, 1, a, a^2, a^3, \cdots$$

and the corresponding exponents

$$(2) \qquad \cdots, -3, -2, -1, 0, 1, 2, 3, \cdots$$

form a geometrical and an arithmetical progression, with the property that to the product of any two numbers of the former corresponds the sum of the corresponding numbers of the latter. Moreover, this property is readily seen to hold for any geometrical progression and any arithmetical progression, provided the number 1 is in the former and corresponds to the number 0 in the latter. If then any number of geometric means be inserted between two numbers of (1) and the same number of arithmetic means be inserted between the corresponding terms of (2), the two sequences of numbers will still have the fundamental property mentioned. It remains only to show that by carrying

* Cf. C. Färber, *Archiv der Mathematik und Physik*, vol. 15 (July, 1909), p. 75.

this process far enough, any given positive number may be approximated to as closely as we please in the sequence (1).

This is the only place, it should be added, in the two volumes in which it appears that one of Professor Klein's suggested pedagogical reforms is impracticable in this country. The section on the logarithm closes with a consideration of this function from the point of view of the theory of functions of a complex variable. Naturally the exponential function is disposed of in the same connection.

There follows an extended section on the trigonometric functions (and also of the hyperbolic functions) mainly from the point of view of the theory of functions of a complex variable. The applications of the trigonometric functions also receive considerable attention, especially as regards the theory of small oscillations and (more extensively) the representation of an arbitrary function by Fourier series. This third and last main part on analysis closes with a treatment of the infinitesimal calculus as such. The sequence of topics is again historical, the comment critical. Some twenty pages (a little less than half of this section) is devoted to Taylor's theorem. Throughout there is much emphasis on geometric representation; the connection of the calculus with the theory of finite differences and the problems of interpolation is clearly exhibited, and many interesting comments on pedagogical questions are inserted.

The first volume closes with an appendix in which the author proves the transcendental character of e and π, and then gives a beautifully clear account of the fundamental ideas in the theory of sets (Mengenlehre).

In the second volume the author sets himself the problem of giving a survey of the whole field of geometry. As has already been said all pedagogical questions are here reserved for separate discussion in an appendix. The method is almost exclusively analytic, the applications are very largely in the field of mechanics. As in the first volume, there are also here three main parts: Part I: The simplest geometric forms; Part II: Geometric transformations; Part III: Systematic development and the foundations of geometry.

Part I begins with a discussion of the well known expressions of length, area, and volume as determinants, in which the significance of the algebraic sign of these expressions receives particular attention. This is followed by a detailed discussion of Grassmann's systematic use of determinants for the definition

of geometric quantities. We have here a very readable exposition of the comparatively little known ideas that Grassmann developed in his Ausdehnungslehre. The notions here developed are at once coordinated with fundamental concepts in mechanics. Throughout also the history of the subject is emphasized. Vector analysis, Plücker's line coordinates, n dimensional geometry are among the topics introduced. As a detail we may mention that Poncelet is given the whole credit as the discoverer of the principle of duality, while Gergonne, who would seem to deserve as much credit in this discovery, is not even mentioned.

Part II discusses in order and with considerable detail the affine and the projective transformations, with many interesting applications to geometry and in particular to the problems of perspective drawing. Here the author makes a strong plea that every teacher of mathematics should be familiar with the principles of descriptive geometry, a plea with which the reviewer is in full sympathy. This is followed by a briefer discussion of higher transformations, the transformations by reciprocal radii, and the several methods in use in the construction of geographical maps receiving special attention. The most general continuous point transformations are then briefly considered in connection with problems of analysis situs. After a section of less than twenty pages devoted to the dualistic and contact transformations, the author devotes some thirty pages to the introduction and use of imaginary elements into geometry, von Staudt's classical work in this respect receiving detailed consideration.

Part III, finally, itself falls into two parts. The first of these is devoted to the systematic organization of the field of geometry. Klein's well-known use of the group concept as a means of geometric classification is of course fundamental. On the analytic side the geometry associated with a given group then becomes the theory of invariants for this group. We are, therefore, given an extended survey of this theory and its applications to geometry. This section closes with the special developments regarding the characterization of the affine and metric geometries within the general projective geometry. The second part is devoted to the logical foundations of geometry. In many respects this is the most interesting part of the volume; in one fundamental particular, however, it is the least satisfying. The discussion of the two methods of building up

the ordinary metric geometry on the one hand by making the group of rigid displacements fundamental, on the other by choosing as undefined the notions of distance, angle, and congruence is beautifully clear. The discussion of the foundations as they appear in Euclid's Elements that follows, with its critical comment and the clear description in historical sequence of the developments to which the problems involved gave rise, forms a remarkably stimulating bit of reading.

What appears to us as unsatisfactory in Professor Klein's treatment is due to his point of view. This applies as well to his treatment of the foundations of analysis. This point of view is summed up in the statement that Professor Klein can see no virtue in the purely formal and abstract treatment of these questions. He expresses himself very clearly on this point on several occasions. To regard the undefined elements as mere symbols devoid of meaning (except such as is implied in the explicit assumptions concerning them) and to regard the unproved propositions as mere arbitrary assumptions appears to Professor Klein as "the death knell of all science" (volume 2, page 384). The reviewer is obliged to take emphatically a diametrically opposite view. We should say this formal point of view has given to science a new and powerfully vital principle. It is without doubt, so it seems to us, the most powerful unifying principle in our science today. We will try briefly to justify these assertions. The formal logical consequences of a set of assumptions concerning certain undefined symbols constitute what we will here call an *abstract theory*. If any concrete meaning is assigned these symbols for which all the assumptions appear to be satisfied, we obtain what we will call a *concrete representation* of the abstract theory. From the point of view of the foundations such a concrete representation is regarded as a consistency proof of the assumptions. We are in full accord with what Professor Klein has to say as to the logical limitations of such a "proof" and the important metaphysical problems to which the formal point of view gives rise (volume 1, pages 33–36); we are glad that he has insisted on these matters so emphatically. However, our present interest is in *the application of the formal treatment outside of the field of the logical foundations*. A given abstract theory may have many different concrete representations. These different concrete representations are then unified by the underlying abstract theory. From this point of view geometry and analysis are coex-

tensive, a fact which in a course of lectures of the kind here considered we might have expected to see strongly emphasized.

Professor Klein would probably take issue with us right here. To him geometry is essentially intuitional (anschaulich); if it is not intuitional it is not geometry, he would probably say. That he holds this point of view is abundantly shown. He insists for example that geometric methods are less precise and rigorous than analytic methods, a statement which is certainly true only under the crude intuitional view of geometry. But this is only a quibble of words. Professor Klein is of course justified in using the word geometry to apply only to that which is "anschaulich" (he does speak of n dimensional geometry, however), if he so desires. Whatever we call it, however, there is such a thing as an abstract theory underlying geometry, which to avoid circumlocution we call geometry. The methods of this abstract geometry are just as precise and rigorous as those of (abstract) analysis, no more and no less. The unifying power of this point of view, to mention only one example, is seen in the fact that projective geometry and certain tactical configurations (triple systems, etc.), with only a finite number of elements appear as concrete representations of the same abstract theory. The formal point of view is of the greatest advantage also in the consideration of imaginary elements in geometry. After describing von Staudt's method of introducing complex elements into geometry, whereby certain involutions are taken to represent complex points and calling attention to the fact that the whole theory of complex projective geometry could be built up on this basis, Professor Klein remarks (volume 2, page 270): "In most cases, however, the application of this geometric interpretation (of complex elements) would, in spite of its theoretical advantages, be attended with so many complications that one may well be satisfied with its theoretical possibility and that one will return to the naiver point of view to the effect that a complex point is a set of imaginary coordinates with which one may in a certain way operate as with the real points." From the abstract point of view a complex point is just the same sort of a point in complex geometry as a real point; in fact this distinction is entirely meaningless, until a chain has been selected as the real chain. It is unnecessary to multiply examples. The advantages of a formal treatment are briefly expressed as follows: If an abstract theory is developed concerning a set of mere symbols, then this

theory applies to *every* concrete representation of these symbols for which the fundamental assumptions are satisfied. These concrete representations of a given abstract theory may be many and varied in character; the abstract theory serves to unify them all.

It would seem that a clear understanding of this point of view is of particular importance to the prospective teacher. It is now well recognized that a knowledge of the foundations of mathematics is essential in the best preparation of the teacher, and the abstract point of view, if not absolutely necessary, greatly facilitates a clear understanding of the problems here presented. That the non-euclidean geometries serve, though not as conveniently, to describe the properties of our intuitional space, is merely due to the fact that the points, lines, etc., of the latter may be regarded as satisfying all the assumptions lying at the basis of each of these geometries.

The volume closes with a very interesting account of the movements toward reform in the teaching of elementary geometry as they have developed during the past years in England, France, Italy, and Germany. The work as a whole is a remarkable example of the distinguished author's mastery of the art of clear and stimulating exposition. We sincerely hope that it will have a wide influence, also in America, in arousing an active interest in a more serviceable preparation of our teachers.

J. W. YOUNG.

CORRECTION.

The following misprints occur in page 122 of Dr. Onnen's paper in the December number of the BULLETIN:

Lines 4–5. *For* . . . dividing n times by any integer a . . . *read* . . . dividing n times by a any integer. . . .

Line 6 from the bottom. *For* . . . an integral value for n . . . *read* . . . an integer. . . .

NOTES.

THE opening (January) number of volume 11 of the *Transactions of the American Mathematical Society* contains the following papers: "Theorems on simple groups," by H. F. BLICHFELDT; "Infinite discontinuous groups of birational

transformations which leave certain surfaces invariant," by V. SNYDER; "Proper multiple integrals over iterable fields," by E. B. LYTLE; "On a class of hyperfuchsian functions," by C. F. CRAIG; "Periodic orbits about an oblate spheroid," by W. D. MACMILLAN.

AT the sixty-first meeting of the American association for the advancement of science, held at Boston, December 27, 1909, to January 1, 1910, Professor A. A. MICHELSON was elected president, Professor E. H. MOORE was elected vice president and chairman of Section A, and Professor G. A. MILLER was re-elected secretary of the section. The next meeting of the association will be held at Minneapolis.

AT the meeting of the London mathematical society held on December 9 the following papers were read: By T. H. BLAKESLEY, "Exhibition of an instrument for solving cubic equations;" by A. B. BASSET, "The connection between the theories of the singularities of surfaces and double refraction;" by W. BURNSIDE, "On the representation of a group of finite order as a group of linear substitutions with rational coefficients;" by A. L. DIXON, "The eliminant of the equations of four quadric surfaces."

THE course of lectures on "Graphical methods in mathematics and physics," delivered, 1909–1910, by the Kaiser Wilhelm professor in Columbia University, Professor CARL RUNGE of the University of Göttingen, is to be published in book form by Columbia University. The subject is one which has not received due attention in this country or abroad. A considerable amount of the material contained in the lectures is original with Professor Runge. The methods developed have many important applications in astronomy, physics, engineering, and various departments of technology.

THE firm of Martin Schilling in Halle announces a number of new models: Series XXX, numbers 6 and 7, plaster models of two surfaces of order 12, class 10, applicable to the surface of the paraboloid of revolution, constructed under the direction of Professor Darboux by Dr. E. ESTANAVE; Series XXXIII, numbers 2 and 3, two thread models of the discriminant surface of a quartic equation, constructed at the suggestion of Pro-

fessor Klein and under the direction of Professor F. Schilling by R. HARTENSTEIN; Series XXXV, numbers 1–27, twenty-seven card models of spherical quadrilaterals, constructed with the cooperation of Professor Schilling by Dr. W. IHLENBURG; Series XXVI, numbers 19–23, new models in descriptive, projective, and analytic geometry, constructed by Professor F. SCHILLING. The last series consists of spherical blackboards of various sizes, together with mountings and movable rings, and two one-sheeted hyperboloids with their asymptotic cones.

THE academy of sciences of Paris has contributed 500 francs towards the expense of erecting a monument to Laplace at Beaumont en Auge, where he was born in 1746.

AT the meeting of the academy of sciences of Paris on December 6, the Leconte prize (2,000 francs) was awarded to M. RITZ for his contributions to mathematical physics and to mechanics.

THE prize of 800 francs announced by the Belgian academy for an important contribution to the differential geometry of ruled space (see BULLETIN, volume 15, page 44) has been awarded to Professor E. J. WILCZYNSKI for his memoir on "The general theory of congruences."

THE December number (volume 18, numbers 11–12) of the *Jahresbericht der Deutschen Mathematiker-Vereinigung* contains a reproduction of the Abel monument which was unveiled at Christiania, October 17, 1908. The total contributions to the cost of the monument amounted to about $10,000, all but $1,000 of which were subscribed in Norway.

THE December number (volume 2, number 2) of *The Mathematics Teacher* contains a report of the organization of the American work of the international commission on the teaching of mathematics, including the names of the 237 members of sub-committees. In Germany this work is assigned to individuals instead of to committees, each person preparing an exhaustive monograph. Besides a number of minor reports of organization and progress, one complete report has already appeared, that by Dr. W. LIETZMANN on "Material and method in mathematical instruction in northern Germany, based upon existing text-books."

DURING the year 1906–1907 the following persons received the degree of doctor of philosophy from the German universities, with mathematics as the major subject. The title of the dissertation is added.

Berlin.

KERL, O. "Voranschläge der Genauigkeit beim trigonometrischen Punkteinschalten."

KNÖPP, K. "Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze."

Breslau.

STEROSTZIK, H. "Ueber eine von Steiner gefundene, noch wenig beachtete Eigenschaft der Leitstrahlen der Kegelschnitte und über Kurven, die mit den einen Kegelschnitt doppelt berührenden Kreisen zusammenhängen."

VOGT, W. "Korrelative Räume bei gegebener Punktkernfläche."

WEISS, E. "Anzahlbestimmungen für das Strahlennetz (lineare Kongruenz)."

WIESING, O. "Ueber eine zwei-zweideutige Verwandtschaft zwischen zwei Ebenen und ihr Analogon im Raume."

Erlangen.

EGERER, H. "Ueber die Curve der Ecken der Vierseite, die von den gemeinsamen Tangenten eines festen Kegelschnitts und der Kegelschnitte eines Büschels gebildet werden."

HAUSER, W. "Ueber Resultanten- und Diskriminantenbildung in der Theorie der elliptischen Thetafunktionen."

Giessen.

HENSEL, G. "Ueber permutable Gruppenbasen aus zwei Elementen."

LANGE, M. "Die Verteilung der Elektrizität auf zwei leitenden Kugeln in einem zu ihrer Zentrallinie symmetrischen elektrostatischen Felde."

THAER, C. "Ueber Invarianten, die symmetrischen Eigenschaften eines Punktsystems entsprechen."

Göttingen.

BORN, M. "Untersuchungen über die Stabilität der elastischen Linie in Ebene und Raum, unter verschiedenen Grenzbedingungen."

BROGGI, U. "Die Axiome der Wahrscheinlichkeitsrechnung."

CRATHORNE, A. R. "Das räumliche isoperimetrische Problem."

GILLESPIE, D. C. "Anwendungen des Unabhängigkeitsgesetzes auf die Lösung der Differentialgleichungen der Variationsrechnung."

HASEMAN, C. "Anwendung der Theorie der Integralgleichungen auf einige Randwertaufgaben in der Funktionentheorie."

LEBEDEFF (Miss), W. "Die Theorie der Integralgleichungen in Anwendung auf einige Reihenentwickelungen."

WILLERS, F. A. "Die Torsion eines Rotationskörpers um seine Achse."

Greifswald.

APFELSTEDT, M. "Ueber eine Gattung von projektiven Transformationsgruppen in fünf Veränderlichen."

BROSZAT, W. "Ueber Scharen von ∞^4 Flächen im R_3, die durch Berührungstranformation in Scharen von ∞^4 Kurven, überführbar sind."

REICHEL, W. "Ueber trilineare alternierende Formen in sechs und sieben Veränderlichen und die durch sie definierten geometrischen Gebilde."

ROELCKE, O. "Ueber die Bäcklundsche Transformation der Flächen konstanter Krümmung."

Heidelberg.

SPEYERER, K. "Ueber Wärmeströmung in dünnen, frei anstrahlenden Platten."

Jena.

DURHOLD, P. "Ueber ein Kreisbündel sechster Ordnung."

Kiel.

HANSEN, O. "Ueber die äquiforme Geometrie im Bündel."

Königsberg.

FOETHKE, E. Anwendung des erweiterten euklidischen Algorithmus auf Resultantenbildungen."

Marburg.

HÜTTIG, F. "Arithmetische Theorie eines Galoischen Körpers."

OETTINGER, E. "Ueber stationäre Gasbewegungen mit Berücksichtigung der inneren Wärmeleitung."

Munich.

DINGLER, H. "Beiträge zur Kenntnis der infinitesimalen Deformation einer Fläche."

FUCHS, F. "Beiträge zur Theorie der elektrischen Schwingungen einer leitenden Rotationsellipsoides."

MÜNICH, K. "Ueber nicht-euklidische Cykliden."

SCHÜBEL, H. "Aufstellung von nicht-euklidischon Minimalflächen."

Münster.

GESSNER, E. "Ueber die Asymptotenkurven einer Schar Konoidflächen im allgemeinen und die des Cylindroids im besonderen."

LANGENKAMP, O. "Ueber Saccheri's Untersuchungen des Parallelenaxioms."

Rostock.

DOLGE, P. "Ueber Bernouillische Zahlen und Funktionen, welche zu einer Fundamentaldeterminante gehören und deren Anwendung auf die Summation unendlicher Reihen."

HELLWIG, M. "Untersuchung über die Lage der Inzidenzpunkte bei Reflexion und Refraktion an Ebene, Kugel und Kreiszylinder für zwei feste Punkte im Raume (Licht- und Augenpunkt)."

Strassburg.

ALTERAUGE, L. "Ueber lineare Relationen zwischen hypergeometrischen Integralen."

ENDERS, M. A. "Ueber die Darstellung der Raumkurve vierter Ordnung vom Geschlecht 1 durch Thetafunktionen."

HUPPERZ, R. W. "Analytische Untersuchung der allgemeinen Schraubenregelfläche. Eine monographische Studie."

MAGENER, F. W. A. "Anallagmatische Punktkoördinaten im Kegelgebüsch und ihre Anwendung auf die nicht-euklidische Geometrie."

RUTHINGER, M. "Die Irreducibilitätsbeweise der Kreisteilungsgleichung."

SAUER, K. "Zur Funktionentheorie auf dem algebraischen Gebilde $s = \sqrt[3]{f_{3n}(z)}$."

SCHUMACHER, H. A. "Ueber eine Riemann'sche Funktionenklasse mit zerfallender Thetafunktion."

ZAHN, K. A. K. "Constructive Bestimmung der Hauptaxen und der Umrisse einer Fläche zweiten Grades, die durch einen Kreis und vier Punkte des Raumes bestimmt ist."

Tübingen.

HOFFMANN, G. "Das Abel'sche Theorem für die elliptischen Integrale."

LÜFFLER, E. "Beiträge zur Theorie der Schnittpunkte algebraischer Kurven."

For the year 1907–1908 the list is as follows.

Berlin.

SCHEE, W. "Ueber irreguläre Potenzreihen und Dirichletsche Reihen."

Bonn.

KUMMER, A. "Ueber eine Gattung von projektiven Transformationsgruppen in sechs Veränderlichen."

Breslau.

SCHMIDT, R. "Ueber zweite Polarflächen einer allgemeinen Fläche 4. Ordnung."

Erlangen.

NOETHER, (Miss) E. "Ueber die Bildung des Formensystems der ternären biquadratischen Form."

Giessen.

KEINEMANN, E. "Die gemischte Kegelschnittschar (das Kegelschnittsystem $s(1p, 3l)$)."

KÜBEL, K. G. J. "Anwendungen einer anschaulichen Darstelluug des Imaginären."

JENZ, O. "Ueber singuläre Asymptotenkurven."

Göttingen.

CAIRNS, W. D. W. "Die Anwendung der Integralgleichungen auf die zweite Variation bei isoperimetrischen Problemen."

HELLINGER, E. "Die Orthogonalinvarianten quadratischer Formen von unendlich vielen Variabeln."

KÖNIG, R. "Die Oscillationseigenschaften der Eigenfunktionen der Integralgleichung mit definitivem Kern und das Jacobische Kriterium der Variationsrechnung."

LAUMANN, T. "Ueber den Isomorphismus von Gruppen linearer Substitutionen mit reellen und mit komplexen Koeffizienten."

SWIFT, E. "Ueber die Form und Stabilität gewisser Flüssigkeitstropfen."

WEYL, H. "Singuläre Integralgleichungen mit besonderer Berücksichtigung des Fourierschen Integraltheorems."

Halle.

BRANDES, H. "Ueber die axiomatische Einfachheit mit besonderer Berücksichtigung der auf Addition beruhenden Zerlegungsbeweise des Pythagoräischen Lehrsatzes."

ERFURTH, P. "Die Komplementärflächen der pseudosphärischen Schraubenflächen."

MAHLO, P. "Topologische Untersuchungen über Zerlegung in ebene und sphärische Polygone."

MORGENSTERN, A. "Beiträge zur numerischen Lösung der Gleichung fünften Grades."

ZÖLLICH, H. "Beiträge zur Theorie der ganzen transzendenten Funktionen der Ordnung Null."

Jena.

HEILAND, F. "Hüllflächen einer Schar von Regelflächen zweiter Ordnung."

Kiel.

DIECK, W. "Zur Klassifikation der Punktepaare und Kegelschnittbüschel."

GURSKI, V. "Ueber den Zusammenhang zwischen den partikulären Lösungen der einzelnen Gebiete bei der hypergeometrischen Differentialgleichung dritter Ordnung mit zwei endlichen singulären Punkten."

HASS, P. "Zur Definition des Begriffs der eindeutigen analytischen Funktion."

Königsberg.

ARNDT, B. "Ueber die Verallgemeinerung des Krümmungsbegriffs für Raumkurven."

DORNER, O. "Ueber Teiler von Formen."

JANZEN, O. "Ueber einige stetige Kurven, über Bogenlänge, linearen Inhalt und Flächeninhalt."

KALUZA, T. "Die Tschirnhaustransformation algebraischer Gleichungen mit einer Unbekannten."

KISCHKE, R. "Ueber Fehlerabschätzung bei unendlichen Produkten und deren Anwendungen."

Leipzig.

SCHILLER, G. "Die Bewegung einer homogenen Kugel auf einer materiellen Parabel unter dem Einfluss der Schwerkraft."

Munich.

CRAMER, F. H. "Ueber die Erniedrigung des Geschlechts abelscher Integrale, insbesondere elliptischer und hyperelliptischer, durch Transformation."

HORN, C. "Konforme Abbildung eines von gewissen Kurven begrenzten Flächenstücks auf den Einheitskreis."

MOHRMANN, H. "Beiträge zur Theorie der Singularitäten des algebraischen Linien-Complexes beliebigen Grades."

WALEK, K. "Binäre kubische Transformation und Complexe."

Münster.

KREFT, W. "Beiträge zur Goursatschen Transformation der Minimalflächen."

Rostock.

PEECK, H. "Ein Beitrag zur Theorie der gebrochenen Fokaldistanzen."

WEISSE, E. "Anwendungen der elliptischen Funktionen auf ein Problem aus der Theorie der Gelenkmechanismen."

Strassburg.

BRAND, E. L. "Ueber Tetraëder, deren Kanten eine Fläche zweiter Ordnung berühren."

BRESSLAU, H. S. S. "Dirichlet's Satz von der arithmetischen Reihe für den Körper der dritten Einheitswurzeln."

KEMPF, A. "Tetraëder, deren Kanten eine Fläche F_2 zweiter Ordnung berühren."

Tübingen.

KÖSTLIN, E. "Ueber eine Deutung der Gleichung, die wischen dem Bogen und dem Neigungswinkel der Tangente zm Endpunkt des Bogens einer ebenen Kurve besteht."

REIFF, —. "Rollen einer Kugel in einem Zylinder ohne Einwirkung der Schwerkraft."

SCHWARTZ, R. "Der Eisensteinsche Satz über die Koefficienten der Reihenentwicklungen algebraischer Funktionen."

THE mathematical society of Berlin announced a special session to be held January 8, 1910, to commemorate the centenary of the birth of E. E. KUMMER. The principal address was given by Professor K. HENSEL, of the University of Marburg.

THE Göttingen academy of sciences announces that it has made an award of 100 Marks from the interest of the Wolfskehl foundation to Dr. A. WIEFERICH, of Münster, who has succeeded in proving that the equation $x^p + y^p = z^p$ cannot be solved in terms of positive integers, not multiples of p, if $2^p - 2$ is not divisible by p^2. (*Crelle's Journal*, volume 137.) "This surprisingly simple result represents the first advance since the time of Kummer in the proof of the last Fermat theorem."

PROFESSOR M. J. M. HILL, of the University of London, has been elected honorary fellow of St. Peter's College, Cambridge.

DRS. A. N. WHITEHEAD and H. F. BAKER have been appointed chairmen of examiners for parts I and II, respectively, of the mathematical tripos for 1910.

AT the annual meeting of the Royal society of London on November 30, the Copley medal was conferred upon Dr. G. W. HILL, of West Nyack, New York, and a royal medal upon Professor A. E. H. LOVE, of Oxford University.

ON the occasion of the dedication of the new buildings at the University of Stockholm on December 16 honorary doctorates were conferred upon Professors H. POINCARÉ and P. PAINLEVÉ, of the University of Paris.

AT the seventy-fifth anniversary of the University of Brussels on November 27, the degree of doctor of laws was conferred upon Professor H. POINCARÉ, of the University of Paris.

DR. H. C. MCWHEENEY has been appointed professor of mathematics at University College, Dublin.

PROFESSOR R. SALIGER, of the German technical school at Prague, has been appointed professor of mechanics at the technical school at Vienna.

PROFESSOR L. SCHLESINGER, of the University of Klausenburg, has been elected to membership in the academy of sciences of Halle.

PROFESSOR G. KOWALEWSKI, of the University of Bonn, has been appointed professor of mathematics at the German technical school at Prague.

DR. A. TIMPE has been appointed docent in mechanics at the technical school at Aachen.

DR. TH. KALUZA has been appointed docent in mathematics at the University of Königsberg.

PROFESSOR G. BOCCARDI, of the University of Turin, has been promoted to a full professorship of mathematical astronomy.

PROFESSOR C. SEVERINI, of the University of Catania, has been promoted to a full professorship of analytic geometry.

PROFESSOR T. BOGGIO, formerly of the University of Messina, then at the Institute at Florence, has been appointed associate professor of rational mechanics at the University of Turin.

PROFESSOR E. SOLER, formerly of the University of Messina, has been appointed to a full professorship of theoretical geodesy at the University of Padua.

PROFESSOR M. ABRAHAM, of the University of Göttingen, has been appointed professor of rational mechanics at the technical school of Milan.

DR. U. SCARPIS has been appointed docent in algebraic analysis at the University of Bologna.

DR. E. LAURA has been appointed docent in rational mechanics at the University of Turin.

DR. F. SIBIRANI has been appointed docent in the calculus at the University of Bologna.

THE degree of LL.D. has been conferred by Columbia University on the Kaiser Wilhelm Professor, CARL RUNGE.

PROFESSOR C. J. KEYSER has been appointed head of the department of mathematics at Columbia University to succeed Professor J. H. VAN AMRINGE on the latter's retirement at the close of the present academic year.

PROFESSOR A. S. CHESSIN will deliver at the University of Pennsylvania in February a course of lectures on "The gyrostat and its modern applications."

PROFESSOR E. MILLER, of the University of Kansas, will retire from active service at the close of the present academic year under the conditions of the Carnegie foundation.

NEW PUBLICATIONS.

(*In order to facilitate the early announcement of new mathematical books, publishers and authors are requested to send the requisite data as early as possible to the Departmental Editor,* PROFESSOR W. B. FORD, *1345 Wilmot Street, Ann Arbor, Mich.*)

I. HIGHER MATHEMATICS.

AUTONNE (L.). Sur les groupes de matrices linéaires non invertibles. Paris, 1909. 8vo. 80 pp. Fr. 5.00

BECK (H.). See BÔCHER (M.).

BLUMENTHAL (O.). Principes de la théorie des fonctions entières d'ordre infini. Paris, Gauthier-Villars, 1909. 8vo. 7 + 150 pp. Fr. 5.50

BÔCHER (M.). Einführung in die höhere Algebra. Deutsch von H. Beck. Mit einem Geleitwort von E. Study. Leipzig, Teubner, 1910. 8vo. 12 + 348 pp. Cloth. M. 7.00

BOLZA (O.). Vorlesungen über Variationsrechnung. Umgearbeitete und stark vermehrte deutsche Ausgabe der "Lectures on the calculus of variations" desselben Verfassers. 3te Lieferung. Leipzig, Teubner, 1909. 8vo. 9 pp. + pp. 541–705. M. 5.00
In 1 volume, cloth. M. 20.00

BOREL (E.). Leçons sur la théorie de la croissance, professées à la Faculté des Sciences de Paris, recueillies et rédigées par A. Denjoy. Paris, Gauthier-Villars, 1909. 8vo. 6 + 172 pp. Fr. 5.50

BRIOSCHI (F.). Opere matematiche, pubblicate per cura del comitato per le onoranze a F. Brioschi. Vol. V (ultimo). Milano, Hoepli, 1909. 4to. 12 + 556 pp. L. 30.00

BÜCHER, neue, über Naturwissenchaften und Mathematik. (Die Neuigkeiten des deutschen Buchhandels nach Wissenschaften geordnet.) Mitgeteilt Herbst 1909. Leipzig, Hinrich. 8vo. Pp. 49–67. M. 0.30

DENJOY (A.). See BOREL (E.).

DETTE (W.). Analytische Geometrie der Kegelschnitte. Leipzig, Teubner, 1909. 8vo. 6 + 232 pp. Cloth. M. 4.40

EMDE (F.). See JAHNKE (E.).

GALILEO (G.) Le opere. Edizione nazionale sotto gli auspicî di Sua Maestà il Re d'Italia. Vol. XX (ultimo). Firenze, Barbèra. 1909. 4to. 589 pp.

GANS (R.). Einführung in die Vektoranalysis mit Anwendungen auf die mathematische Physik, 2te Auflage. Leipzig, Teubner, 1909. 8vo. 10 + 126 pp. Cloth. M. 3.60

GRAF (J. H.). Einleitung in die Theorie und Auflösung der gewöhnlichen Differentialgleichungen nebst vielen Uebungsbeispielen. Bern, Wyss, 1910. 8vo. 6 + 115 pp. M. 2.00

GRUHN (P.). Mathematische Formelsammlung. Hannover, Jänecke, 1909. 8vo. 62 pp. M. 1.20

GUYGOU (E.). Notes sur les approximations numériques. 3e édition. Paris, 1909. 8vo. 32 pp. Fr. 1.00

IGNATOWSKY (W. VON). Die Vektoranalysis und ihre Anwendung in der theoretischen Physik. (2 Teil.) Teil I: Die Vektoranalysis. Leipzig, 1909. 8vo. 128 pp. M. 2.60

JAHNKE (E.) und EMDE (F.). Funktionentafeln mit Formeln und Kurven. Leipzig, 1909. 8vo. 12 + 176 pp. Cloth. M. 6.00

JOPKE (A.). Synthetische Untersuchungen über lineare Kegelschnittsysteme 1ster, 2ter und 3ter Stufe. (Diss.) Breslau, 1909. 8vo. 56 pp. M 1.80

KARST (L.). Lineare Funktionen und Gleichungen. Lichtenberg, 1909. 4to. 44 pp. M. 1.50

KIEPERT (L.). Grundriss der Differential- und Integralrechnung. 1ter Teil: Differentialrechnung. 11te unveränderte Auflage des gleichnamigen Leitfadens von M. Stegemann. Hannover, Helwing, 1910. 8vo. 20 + 818 pp. M. 12.50

KLIEM (F.). Ueber Oerter von Treffgeraden entsprechender Strahlen in eindeutig und linear verwandten Strahlengebilden 1ster bis 4ter Stufe. (Diss.) Breslau, 1909. 8vo. 52 pp. M. 1.50

KOMMERELL (V. und K.). Allgemeine Theorie der Raumkurven und Flächen. 1ter Band, 2te Auflage. (Sammlung Schubert, XXIX.) Leipzig, Göschen, 1909. 8vo. 8 + 172 pp. Cloth. M. 4.80

LANDAU (E.). Handbuch der Lehre von der Verteilung der Primzahlen. Leipzig, Teubner, 1909. 8vo. Vol. I, 18 + 564 pp. M. 20.00
Vol. II, 9 pp. + pp. 565–961. M. 14.00

LINDEMANN (F.). Ueber den sogenannten letzten Fermatschen Satz. Leipzig, Veit, 1909. 8vo. 5 + 82 pp. M. 3.50

PICKFORD (A. G.). Elementary projective geometry. Cambridge, 1909. 8vo. Cloth. 4s.

RIQUIER (C.). Les systèmes d'équations aux dérivées partielles. Paris, Gauthier-Villars, 1910. 8vo. 23 + 590 pp. Fr. 20.00

SCHIEDE (J.). Die Begriffe der Funktion und des Differentialquotienten in der Gymnasialprima. Köslin, 1909. 4to. 31 pp. M. 1.50

SERRET (J. A.). Cours d'algèbre supérieure. 6e édition. Vols. I–II. Paris, Gauthier-Villars, 1910. 8vo. Vol. I, 13 + 648 pp. Vol. II, 12 + 694 pp. Les deux volumes. Fr. 25.00

STARKE (P.). Planimetrische Konstruktionsaufgaben mit Verwendung der harmonischen Teilung und des Apollonischen Kreises. Leipzig, 1909. 4to. 71 pp.

STEGEMANN (M.). See KIEPERT (L.).

STUDY (E.). See BÔCHER (M.).

STURM (R.). Die Lehre von den geometrischen Verwandtschaften. 4ter Band: Die nichtlinearen und die mehrdeutigen Verwandtschaften zweiter und dritter Stufe. (B. G. Teubner's Sammlung, XXVII, 4.) Leipzig, Teubner, 1909. 8vo. 10 + 486 pp. Cloth. M. 20.00

THIELE (T. N.). Interpolationsrechnung. Leipzig, Teubner, 1909. 8vo. 12 + 175 pp. M. 10.00

WELLISCH (S.). Theorie und Praxis der Ausgleichungsrechnung. 1ster Band: Elemente der Ausgleichungsrechnung. Mit einem Bildnisse von K. F. Gauss. Wien, Fromme, 1909. 8vo. 11 + 276 pp. M. 10.00

WILLOT (J.). Exposé des fondements de la géométrie. Paris, Gauthier-Villars, 1908. 8vo. 53 pp. Fr. 1.75

II. ELEMENTARY MATHEMATICS.

BARNARD (S.) and CHILD (J. M.). A new algebra. Parts 1-4 with answers. London, Macmillan, 1909. 8vo. 544 pp. Cloth. 4s.

CHILD (J. M.). See BARNARD (S.).

EDGETT (G. C.). Exercises in geometry. Boston, Heath, 1909. 12mo. 6 + 81 pp. Cloth. $0.40

FABRIS (V.) Nozioni elementari di algebra, ad uso della terza classe delle scuole tecniche e della prima classe delle scuole normali. 7a edizione riveduta e corretta dall'autore. Torino, Botta, 1909. 8vo. 39 pp. L. 0.80

FENKNER (H.). Lehrbuch der Geometrie für den Unterricht an höheren Lehranstalten. Ausgabe B. Für Realschulen. (In 2 Teilen.) 2ter Teil: Raumgeometrie und Trigonometrie. Nebst einer Aufgabensammlung. Berlin, Salle, 1910. 8vo. 4 + 88 pp. M. 1.40

FUCINI (C.). Algebra elementare, per le scuole secondarie inferiori. 3a edizione. Genova, Gioventù, 1910. 8vo. 79 pp.

GAUSS (F. G.). Die Teilung der Grundstücke insbesondere unter Zugrundelegung rechtwinkliger Koordinaten. Nebst vierstelligen logarithmischen und trigonometrischen Tafeln, einer Quadrattafel, sowie einer Multiplikations- und Divisionstafel. 5te Auflage. 2 Teile. Berlin, Decker, 1909. 8vo. 195 + 80 pp. Cloth. M. 7.60

HOVESTADT (H.). See KILLING (W.).

HOWARD (H. E.). Easy practical mathematics. London, Longmans, 1909. 8vo. 1s.

JENSON (O.). See WALTHER (F.).

KAMBLY und ROEDER. Planimetrie, neu bearbeitet von A. Thaer. Ausgabe B. Für Realanstalten und Gymnasien mit mathematischem Reformunterricht. 148–151ste Auflage der Kamblyschen Planimetrie. Breslau, Hirt, 1909. 8vo. 240 pp. M. 2.50

KILLING (W.) und HOVESTADT (H.). Handbuch des mathematischen Unterrichts. 1ter Band. Leipzig, Teubner, 1910. 8vo. 8 + 456 pp. Cloth. M. 10.00

KUTNEWSKY (M.). See MÜLLER (H.).

LEMBCKE (K.). Allgemeine Arithmetik und Algebra in ihrer Beziehung zu einander und zu den höheren bürgerlichen Rechnungsarten. 2te, verbesserte Auflage. Wismar, Hinstorff, 1910. 8vo. 12 + 189 pp. M. 3.00

MAHLERT (A.). See MÜLLER (H.).

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